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Abstract

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ON THE SOLVABILITY OF THE FIRST BOUNDARY-VALUE PROBLEM FOR ELLIPTIC EQUATIONS

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In this work the first boundary-value problem for strongly elliptic systems of differential equations is considered. For simplicity we restrict ourselves to a single equation, although all the results remain valid for systems. As is known ([2]), the first boundary-value problem for strongly elliptic systems is solvable if the measure of the domain is sufficiently small. This solvability condition is not sharp, since there exist domains of very large area for which the first boundary-value problem is solvable. In the present work another solvability condition is given; in order to formulate it one needs the notion of capacity associated with an equation of order $2m$; for $m = 1$ this capacity is equivalent to the classical one ([1]). In addition, the notion of capacity proves useful in studying the question of in what sense the generalized solution of the first boundary-value problem satisfies the boundary conditions.

Definition. Let in n-dimensional space there be a cube H with edge h, and let E be a closed set in it. By the **capacity** $e^n_{m,H}(E)$ we shall mean

$$\inf \frac{\iint_H \sum_{i=1}^{m} (D^i u)^2 \, dv}{\iint_H u^2 \, dv},$$

where $D^i u$ denotes all possible partial derivatives of the function u of order i, and the inf is taken over all functions differentiable continuously m times and vanishing on E.

We indicate some of the simplest properties of capacity:

1. If the capacity $e^n_{m,H}(E) = 0$ for some H, then it is equal to zero for every H.

2. If the edge of the cube H tends to ∞, then $e^n_{m,H}(E) \to 0$.

3. If $E_1 \supset E$, then $e^n_{m,H}(E_1) \geq e^n_{m,H}(E)$.

4. If $2m > n$ and $e^n_{m,H}(E) = 0$, then the set E is empty. This property is a consequence of the embedding theorem ([3]).
5. If $2m \le n$ and $e^n_{m,H}(E) = 0$, then the projection of E onto any $(n-2m+1)$-dimensional hyperplane has Lebesgue measure zero.

Let us outline the proof of assertion 5. We shall use the following easily proved fact: if in an n-dimensional cube H the set E has positive n-dimensional Lebesgue measure, then every differentiable function u satisfies the inequality

$$\iint_H u^2\,dv < C\left[\iint_E u^2\,dv + \iint_H \operatorname{grad}^2 u\,dv\right], \tag{*}$$

where the constant C does not depend on the function u.

To prove assertion 5, it is enough to show that the $(n-2m+1)$-dimensional measure of the set E', which is the projection of E onto the $(n-2m+1)$-dimensional face H, is equal to zero.

Through each point of E' draw a $(2m-1)$-dimensional hyperplane orthogonal to the $(n-2m+1)$-dimensional face on which E' lies. On each such hyperplane T there is a point where $u = 0$, and, by property 4, the inequality is valid

$$\iint_{T_1} u^2\,ds \le C_1 \iint_{T_1} \sum_{i=1}^{m} (D^i u)^2\,ds$$

(here $T_1 = T \cap H$).

Integrating this inequality over E', we obtain

$$\iint_K u^2\,dv \le C_1 m(E') \iint_H \sum_{i=1}^{m} (D^i u)^2\,dv$$

where K is the set-theoretic sum of all T_1 and has positive n-dimensional measure. But then, by inequality $(*)$, we have the inequality

$$\iint_H u^2\,dv \le C_2 \iint_H \sum_{i=1}^{m} (D^i u)^2\,dv,$$

which means that the capacity of E is positive; the contradiction obtained shows that $m(E') = 0$, as was required to prove.

Consider the first boundary-value problem for the elliptic equation of order $2m$

$$Lu = 0. \tag{1}$$

If it is unsolvable in a domain G, then there exists a nonzero solution of equation (1) with zero boundary conditions.

We shall show that

$$\iint \sum_{i=1}^{m} (D^i u)^2 \, dv \leq K_1 \iint u^2 \, dv, \tag{2}$$

where the constant K_1 depends only on the coefficients of the equation, and the integration is carried out over the whole space. Here it is assumed that $u \equiv 0$ outside G.

Let L_0 be the principal part of L. Then, by the finiteness of u and the ellipticity of L, the inequality holds

$$\iint \sum_{i=1}^{m} (D^i u)^2 \, dv \leq K c_1 \left| \iint (L_0 u) u \, dv \right| + \frac{1}{c_1} \iint u^2 \, dv$$

(c_1 is any number, K depends on m and n). It is easily proved by applying the Fourier transform. By integration by parts it is easy to obtain that

$$\left| \iint ((L - L_0)u) u \, dv \right| \leq \bar{K} c_2 \iint \sum_{i=1}^{m} (D^i u)^2 \, dv + \frac{1}{c_2} \iint u^2 \, dv$$

(c_2 is any number, and $\bar{K}$ depends on the coefficients of (1)).

By virtue of (1),

$$\left| \iint ((L - L_0)u) u \, dv \right| = \left| \iint (L_0 u) u \, dv \right|.$$

Hence

$$K_1 c_2 \iint \sum_{i=1}^{m} (D^i u)^2 \, dv + \frac{1}{c_2} \iint u^2 \, dv \geq \frac{1}{K c_1} \iint \sum_{i=1}^{m} (D^i u)^2 \, dv - \frac{1}{c_1^2} \iint u^2 \, dv.$$

Choosing c_1 and c_2 in the required way, we obtain (2). Divide the whole space into cubes with side length h. At least for one of them, denote it by H, the inequality holds:

$$\iint_H \sum_{i=1}^{m} (D^i u)^2 \, dv \leqslant K_1 \iint_H u^2 \, dv,$$

but this, by the definition of capacity, means that the capacity of the complement of g in the cube H has capacity less than K_1. Thus, if from the cube H one removes a set of capacity less than K_1, then it will be contained in g.

Definition. We shall say that the **inner diameter of the domain** g is less than d with accuracy up to K_1, if there exists a cube with side d which will be contained in g, provided one removes from it a set of capacity less than K_1.

Theorem 1. *If in the elliptic equation*

$$\sum_{k=1}^{2m} \sum_{i_1+\dots+i_n=k} A^{(k)}_{i_1\dots i_n} \frac{\partial^k u}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} + Bu = 0$$

the coefficients satisfy the conditions:

$$(-1)^{m+1} B < b, \qquad \left| D^p A^k_{i_1\dots i_n} \right| < \alpha, \qquad p = 0, 1, \dots, k-1$$

and the inner diameter of the domain g is less than h with accuracy up to K_1, then the first boundary-value problem for such an equation is solvable.

The constants h and K depend only on b and α.

Let us note some consequences that can be obtained from this theorem:

Corollary 1. If $2m > n$ and the inner diameter of g is small in the usual sense, i.e., there exists a number $d > 0$, depending only on the coefficients of the equation, such that the diameter of any sphere contained in g is less than d, then the first boundary-value problem is solvable.

Corollary 2. If $2m = n$, the domain g is simply connected and its inner diameter is small in the usual sense, then the first boundary-value problem is solvable.

As examples show, smallness of the inner diameter in the usual sense is, generally speaking (except for these cases), insufficient for solvability of the first boundary-value problem.

Thus, if one considers the Dirichlet problem for the equation $\Delta u + u = 0$ in the plane, then it may be unsolvable, although the inner diameter of the domain is arbitrarily small. This will be the case if the domain g is a disk of radius π, from which sufficiently many nonintersecting and sufficiently small disks have been removed. Such a domain, of course, is not simply connected. If it were simply connected, then, by Corollary 2, we would obtain that the Dirichlet problem is solvable. If one considers the same equation in three-dimensional space, then the Dirichlet problem may be unsolvable even for a simply connected domain with arbitrarily small inner diameter. An example of such a domain is the following: from a ball of radius π remove so many nonintersecting straight-line segments with one endpoint on the surface of the ball that the inner diameter of the remaining domain is less than any prescribed number, and then replace these straight lines by sufficiently narrow cylinders, upon removing which simple connectedness is preserved. Analogous examples can be constructed for an equation of any order and in a space of any number of dimensions.

We shall now consider the first boundary-value problem for equation (1) in an arbitrary open domain. If no smoothness of the boundary is required, such a problem must be posed in a generalized sense:

Find a generalized solution u of equation (1) such that $u - v$ is a limit in W_2^m of functions that vanish in a boundary strip (or $u - v \in \overset{0}{W}{}_2^m$). Such a solution can be obtained by functional methods ([2]). The question arises in what sense it will satisfy the boundary condition, i.e., in what sense $u - v$ and its differentials are equal to zero on the boundary. To answer this question we introduce the notion of a k-regular point.

Definition. A boundary point A is called ***k*-regular** if

$$\lim_{h \to 0} \frac{e^k(D)}{h^2} > 0,$$

where $e^k(\overline{D})$ is the capacity of the complement of D in the square of side h with center at A.

Theorem 2. *If the point A is k-regular and $u \in \overset{0}{W}{}_2^m$, then the first $(k-1)$ differentials are equal to zero at the point A in the asymptotic sense* $(k = 1, 2, \ldots)$.

As is known, the p-th differential of u is zero at the point A in the asymptotic sense if one can exclude a set E such that the limit of the ratio of the measure of the part of E lying in the square of side a with center at A to a^n is equal to zero as $a \to 0$, and on the remaining set the function u is p times differentiable and its p-th differential is equal to zero at the point A. The result of Theorem 2 for the i-th differential remains valid if, in this definition, one takes such a set E that the limit of the ratio of the measure of the projection of the part of E lying in the square of side a onto any $(n-k+i)$-dimensional hyperplane to a^{n-k+i} is equal to zero as $a \to 0$.

We indicate several sufficient conditions for the k-regularity of a point:

1) If $n = k + 1$ and the point A is reachable from outside the domain by a Jordan curve, then it is k-regular (n is the dimension of the space).

2) If the point A is reachable from outside the domain by a cone, then it is k-regular for any k. If $n > k + 1$, then condition 1) is not sufficient.

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CITED LITERATURE

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[2] M. I. Vishik, Matem. sborn., 29 (71), 3, 615 (1951).
[3] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.

Note: Figure translations are in progress. See original paper for figures.