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Abstract

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ON A CERTAIN SCALE OF SPACES

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Let R be a certain space on which a measure $\mu(E)$ is defined (finite or infinite). We shall consider measurable real-valued functions $f(t)$ defined on R. For each such function denote by $n_f(y)$ the measure of the set of all points $t \in R$ at which $|f(t)| > y$. Denote by S the linear set of all summable finite-valued functions. The functional defined on S

$$\|f\|_{S_\alpha} = \int_0^\infty n_f^{1-\alpha}(y)\, dy \tag{1}$$

has all the properties of a norm. The completion of the space S with respect to the norm (1) will be denoted by S_α. It is not difficult to show that S_α consists of all measurable functions for which the integral in (1) is finite, and the norm of these functions is determined by formula (1).

Let $f \in S$. Denote by $C_1^*, C_2^*, \dots, C_n^*$ the rearrangement, in decreasing order, of the absolute values of all nonzero values of the function $f(t)$, and put $C_{n+1}^* = 0$. Let E_k be the set consisting of all points of R for which $|f(t)| \geq C_k^*$. Then

$$\|f\|_{S_\alpha} = \sum_{k=1}^{n} (C_k^* - C_{k+1}^*)\, [\mu(E_k)]^{1-\alpha}. \tag{2}$$

From this formula it is seen that $\|f\|_{S_1} = C_1^* = \operatorname{vrai\,sup} |f(t)|$ and

$$\|f\|_{S_0} = \sum_{k=1}^{n} (C_k^* - C_{k+1}^*)\, \mu(E_k) = \int |f(t)|\, d\mu.$$

Thus the spaces S_0 and S_1 coincide with the spaces L_1 and L_∞. In the case when $R = [0,1]$ and μ is Lebesgue measure, the spaces S_α were considered earlier by the authors (1).

The norm was introduced by the equivalent formula

$$\|f\|_{S_\alpha} = (1-\alpha) \int_0^1 t^{-\alpha} f^*(t)\, dt,$$

where $f^*(t)$ is the rearrangement of the function $|f(t)|$ in decreasing order.

From formula (2) it is seen that the norm $\|f\|_{S_\alpha}$, for $f \in S$, is a logarithmically convex function of α, i.e., for $\alpha < \beta < \gamma$,

$$\|f\|_{S_\beta} \le \|f\|_{S_\alpha}^{\frac{\gamma-\beta}{\gamma-\alpha}} \|f\|_{S_\gamma}^{\frac{\beta-\alpha}{\gamma-\alpha}}. \tag{3}$$

For characteristic functions $\chi_E(t)$ of measurable sets $E \subset R$ of finite measure, inequality (3) becomes an equality.

One of the most important properties of the space S_α is described by the lemma:

Lemma. *Let a seminorm $\Phi(f)$ be defined on the set S such that for any characteristic function $\chi_E(t)$ it is true that*

$$\Phi(\chi_E) \le M\|\chi_E\|_{S_\alpha} = M[\mu(E)]^{1-\alpha}, \tag{4}$$

where M does not depend on the choice of the set E.

Then for any function $f \in S$ the inequality

$$\Phi(f) \le 2^\alpha M\|f\|_{S_\alpha}. \tag{5}$$

holds.

The coefficient 2^α may be omitted if it is known that (4) holds for all functions in S taking only the values $-1, 0, 1$.

Proof. Keeping the notation adopted in (2), we may write

$$f(t) = \sum_{k=1}^{n} (C_k^* - C_{k+1}^*) \operatorname{sign} f(t) \, \chi_{E_k}(t).$$

Let $E_k = E_k^1 + E_k^2$, where $f(t) > 0$ for $t \in E_k^1$ and $f(t) < 0$ for $t \in E_k^2$. Then

$$\begin{aligned}\Phi(f) &\le \sum_{k=1}^{n} (C_k^* - C_{k+1}^*) [\Phi(\chi_{E_k^1}) + \Phi(\chi_{E_k^2})] \\ &\le \sum_{k=1}^{n} (C_k^* - C_{k+1}^*) M\{[\mu(E_k^1)]^{1-\alpha} + [\mu(E_k^2)]^{1-\alpha}\} \\ &\le 2^\alpha M \sum_{k=1}^{n} (C_k^* - C_{k+1}^*) [\mu(E_k)]^{1-\alpha} \le 2^\alpha M\|f\|_{S_\alpha}.\end{aligned}$$

The coefficient 2^α will not appear if

$$\Phi(\operatorname{sign} f(t) \cdot \chi_E(t)) \leq M[\mu(E)]^{1-\alpha}.$$

The lemma is proved.

Corollary 1. Let the measurable function $\varphi(t)$ have the property that for any measurable set $E \subset R$

$$\int_E |\varphi(t)|\, d\mu \leq M[\mu(E)]^{1-\alpha}, \tag{6}$$

where M does not depend on the choice of the set E. Consider the functional

$$\varphi(f) = \int f(t)\varphi(t)\, d\mu. \tag{7}$$

For the seminorm $\Phi(f) = |\varphi(f)|$, by virtue of (6), inequality (4) will hold for all functions in S taking only the values $-1, 0, 1$. It follows from the lemma that $\varphi(f)$ is a functional on S bounded in the norm of the space S_α. Its norm will be computed by the formula

$$\|\varphi\|_{S_\alpha^*} = \sup[\mu(E)]^{\alpha-1} \int_E |\varphi(t)|\, d\mu, \tag{8}$$

where the supremum is taken over all measurable sets $E \subset R$.

It can be shown that for $0 \leq \alpha < 1$ every linear functional on the space S_α is representable in the form (7), where the function $\varphi(t)$ has property (6). Thus, the space S_α^* conjugate to S_α for $\alpha < 1$ may be regarded as the space of all measurable functions $\varphi(t)$ with finite norm (8).

Corollary 2. Let a linear operator A, defined on S and acting into the Banach space F, be bounded in the norm S_α on the set of characteristic functions, i.e. $\|A\chi_E\|_F \leq M\|\chi_E\|_{S_\alpha}$; then it can be extended by continuity to a linear bounded operator acting from S_α into F. To prove the assertion it is enough to apply the lemma to the functional $\Phi(f) = \|Af\|_F$.

Corollary 3. The space S_α is embedded in the space $L_{\frac{1}{1-\alpha}}$, and

$$\|f\|_{L_{\frac{1}{1-\alpha}}} \leq \|f\|_{S_\alpha}. \tag{9}$$

Inequality (9) follows from the fact that the norm $\Phi(f) = \|f\|_{\frac{1}{1-\alpha}}$ on characteristic functions is equal to the norm in the space S_α, and $\Phi(f) = \Phi(|f|)$.

It is easy to see that in the case $\mu(R) < \infty$ the space S_α contains all the spaces $L_{\frac{1}{1-\alpha}+\varepsilon}$ for $\varepsilon > 0$.

Corollary 4. Let three seminorms $\Phi_1(f)$, $\Phi_{1+\tau}(f)$, and $\Phi_2(f)$ be defined on the set S, and suppose that on characteristic functions $\chi_E(t)$

$$\Phi_1(\chi_E) \le M_1 \|\chi_E\|_{S_{\alpha_1}}, \qquad \Phi_2(\chi_E) \le M_2 \|\chi_E\|_{S_{\alpha_2}},$$

and

$$\Phi_{1+\tau}(\chi_E) \le K \Phi_1^{1-\tau}(\chi_E) \Phi_2^{\tau}(\chi_E),$$

where τ is some number between 0 and 1. Then, for $\alpha = \alpha_\tau = (1-\tau)\alpha_1 + \tau\alpha_2$, inequality (5) is valid with the constant $M = K M_1^{1-\tau} M_2^{\tau}$.

Corollary 5 (interpolation theorem). Let $F_1, F_{1+\tau}$, and F_2 be three Banach spaces having a nonempty intersection F, and let, for $h \in F$,

$$\|h\|_{F_{1+\tau}} \le c \|h\|_{F_1}^{1-\tau} \|h\|_{F_2}^{\tau},$$

where c does not depend on the choice of $h \in F$. Let A be an operator defined on S, acting in F, and possessing the properties

$$\|A(\lambda f)\|_{F_k} = |\lambda| \, \|Af\|_{F_k}, \qquad \|A(f+g)\|_{F_k} \le \|Af\|_{F_k} + \|Ag\|_{F_k} \quad (k = 1, 1+\tau, 2).$$

If for characteristic functions $\chi_E(t)$

$$\|A\chi_E\|_{F_1} \le M_1 \|\chi_E\|_{S_{\alpha_1}}$$

and

$$\|A\chi_E\|_{F_2} \le M_2 \|\chi_E\|_{S_{\alpha_2}},$$

then for every $f \in S$ the inequality

$$\|Af\|_{F_{1+\tau}} \le 2^{\tau} c M_1^{1-\tau} M_2^{\tau} \|f\|_{S_{\alpha_\tau}}, \qquad \text{where } \alpha_\tau = (1-\tau)\alpha_1 + \tau\alpha_2. \tag{10}$$

is valid.

Corollary 5 follows from Corollary 4 if one sets $\Phi_k(f) = \|Af\|_{F_k}$ $(k = 1, 1+\tau, 2)$. If the operator A is linear, then from inequality (10) it follows that it can be extended by continuity to a bounded operator acting from the space S_{α_τ} into the space $F_{1+\tau}$.

In the case when the operator A is the identity operator, from inequality (10), under the condition of compatibility of the norms in the spaces $F_{1+\tau}$ and S_{α_τ}, it follows that the space S_{α_τ} can be embedded in the space $F_{1+\tau}$, and our assertion has the character of an embedding theorem.

Definitions. We shall say that an operator A is an **operator of type** (α, β), or of **maximal type** (α, β), or of **minimal type** (α, β) $(0 \leq \alpha, \beta \leq 1)$, if it is a bounded operator acting, respectively, from the space $L_{\frac{1}{1-\alpha}}$ into the space $L_{\frac{1}{1-\beta}}$, or from the space S_α into the space S_β, or from the space $S^*_{1-\alpha}$ into the space $S^*_{1-\beta}$. The spaces $L_{\frac{1}{1-\alpha}}$, S_α, $S^*_{1-\alpha}$ may be defined for functions on R with measure μ, and the spaces $L_{\frac{1}{1-\beta}}$, S_β, $S^*_{1-\beta}$—on another set R_1 with measure μ_1.

Following (^{2}), an operator defined on a functional space will be called **quasilinear** if

$$|A(\lambda, f)| \leq |\lambda|\,|Af|, \qquad |A(f+g)| \leq \varkappa(|Af| + |Ag|). \tag{11}$$

Analogue of the Marcinkiewicz theorem (^{3}). *Let* $0 \leq \alpha_i \leq \beta_i < 1$ $(i = 1, 2)$ *and* $\beta_1 \neq \beta_2$. *If a quasilinear operator* A *has the properties*

$$y\, n_{Af}^{1-\beta_1}(y) \leq M_1 \|f\|_{S_{\alpha_1}} \quad \text{for } f \in S_{\alpha_1}, \qquad y\, n_{Af}^{1-\beta_2}(y) \leq M_2 \|f\|_{S_{\alpha_2}} \quad \text{for } f \in S_{\alpha_2}, \tag{12}$$

then it is an operator of maximal type $(\alpha_\tau, \beta_\tau)$, *where*

$$\alpha_\tau = (1-\tau)\alpha_1 + \tau\alpha_2, \qquad \beta_\tau = (1-\tau)\beta_1 + \tau\beta_2 \quad (0 < \tau < 1).$$

This theorem differs from the known Marcinkiewicz theorem in that, instead of the spaces L_p, the spaces S_α are considered, and instead of operators of type $(\alpha_\tau, \beta_\tau)$, operators of maximal type $(\alpha_\tau, \beta_\tau)$. The proof is carried out according to the same scheme by which A. Zygmund proved the Marcinkiewicz theorem (^{2}). For each function from S_{α_τ} the inequality is established

$$\|Af\|_{S_{\beta_\tau}} \leq K_\tau M_1^{1-\tau} M_2^\tau \|f\|_{S_{\alpha_\tau}} \quad (f \in S_{\alpha_\tau}), \qquad \text{where } K_\tau = \frac{2\varkappa}{|\beta_1 - \beta_2|}\left(\frac{\xi_1}{\tau} + \frac{\beta_2}{1-\tau}\right). \tag{13}$$

When the operator A is linear, the following more general assertion is valid:

Theorem 1. Let $0 \leq \alpha_i \leq \beta_i < 1$ $(i = 1, 2)$, and let $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$. Let the operator A, defined on S, be linear, and let the inequalities

$$y n_{A\chi_E}^{1-\beta_1}(y) \leq M_1 [\mu(E)]^{1-\alpha_1}, \qquad y n_{A\chi_E}^{1-\beta_2}(y) \leq M_2 [\mu(E)]^{1-\alpha_2}. \tag{14}$$

hold for characteristic functions $\chi_E \in S$.

Then, by continuity, it can be extended to an operator of maximal type $(\alpha_\tau, \beta_\tau)$, to an operator of type $(\alpha_\tau, \beta_\tau)$, and to an operator of minimal type $(\alpha_\tau, \beta_\tau)$, where $\alpha_\tau = (1-\tau)\alpha_1 + \tau\alpha_2$, $\beta_\tau = (1-\tau)\beta_1 + \tau\beta_2$ $(0 < \tau < 1)$.

The conditions (14) on characteristic functions coincide with conditions (12); therefore (13) is satisfied on characteristic functions. Then it follows from Corollary 2 that the operator A extends to an operator of maximal type $(\alpha_\tau, \beta_\tau)$. The fact that, under conditions (14), the operator A extends to an operator of type $(\alpha_\tau, \beta_\tau)$ was proved by Stein and Weiss[4]. The possibility of extending the operator A to an operator of minimal type is shown by passing to the adjoint operator.

We note that conditions (14) are certainly satisfied if the operator A is an operator of types (α_i, β_i), or of maximal types (α_i, β_i), or of minimal types (α_i, β_i) $(i = 1, 2)$. Hence it follows:

Corollary. If α_i, β_i $(i = 1, 2)$ satisfy the conditions of the theorem and a linear operator A, defined on S, for $0 < \tau < 1$ extends by continuity to an operator of one of the types $(\alpha_\tau, \beta_\tau)$, then it extends by continuity also to an operator of the two other types $(\alpha_\tau, \beta_\tau)$ $(0 < \tau < 1)$.

For a number of important operators their type is known. This makes it possible to conclude that these operators are simultaneously operators of maximal and minimal type. We give the most important examples.

The operator corresponding to the Hilbert transform on the real axis is an operator of all three types (α, α) for $0 < \alpha < 1$. The operator taking a given trigonometric series into the conjugate one[5] is also an operator of all three types (α, α) $(0 < \alpha < 1)$. The operator of potential type

$$Af = \int \frac{f(y)}{|x-y|^\lambda}\, dy,$$

defined on functions in an n-dimensional domain, for $0 < \alpha < \lambda/n$ is an operator of all three types (α, β), where $\beta = 1 + \alpha - \lambda/n$.

The existence of theorems on operators of potential type in the spaces S_α and S_α^* makes it possible to consider spaces of functions whose generalized derivatives belong to the spaces S_α and S_α^*, and to obtain for them embedding theorems quite analogous to the theorems of S. L. Sobolev[6].

A priori estimates for solutions of elliptic differential equations in the spaces L_p, obtained by A. I. Koshelev[7], can also be regarded as establishing the type of certain operators and as yielding a priori estimates for solutions in the norms S_α and S_α^*.

Finally, we note that the conditions of Theorem 1 are not necessary for the operator A to be simultaneously an operator of the three types $(\alpha_\tau, \beta_\tau)$. It is easy to construct an example of an operator with this property for which condition (14) is not satisfied.

Note added in proof. As became known to the authors, the spaces S_α and their adjoints for the case of Lebesgue measure on $[0,1]$ were studied by Lorentz[8], without connection with interpolation theorems.

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REFERENCES

[1] S. G. Krein, DAN, **132**, No. 3 (1960).
[2] A. Zygmund, J. Math. Pures et Appl., **35**, 223 (1956).
[3] J. Marcinkiewicz, C.R., **208**, 405 (1939).
[4] E. M. Stein, G. Weiss, J. Math. and Mech., Indiana, **8**, No. 2 (1959).
[5] A. Zygmund, *Trigonometric Series*, 1939.
[6] S. L. Sobolev, *Some Applications of Functional Analysis in Problems of Mathematical Physics*, L., 1950.
[7] A. I. Koshelev, UMN, **13**, No. 4 (1958).
[8] G. G. Lorentz, Ann. Math., **51**, No. 1, 37 (1950).

Note: Figure translations are in progress. See original paper for figures.