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Abstract

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ON A SPECIAL INTEGRAL EQUATION ON A CONTOUR OF CLASS R

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The theory of the special integral equation

$$a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\, d\tau = f(t) \tag{1}$$

and of the Riemann problem associated with it, with boundary condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t) \tag{2}$$

on the contour L, was first developed (see ([1,2])) under the following assumptions: 1) the contour L is a simple smooth line, 2) the coefficients $a(t), b(t), f(t), G(t), g(t)$ satisfy a Hölder condition on L.

In subsequent works, extensions were made both of the class of coefficients of these problems ([3–6]) and of the class of contours ([7–9]). In ([7]), the contour L consists of a finite number of piecewise-smooth curves, while in ([8,9]) it is a finite aggregate of curves L_k, for each of which, for some $m > 0$, the inequality $\sigma(t_1, t_2) \leq m|t_1 - t_2|$ holds, where t_1 and t_2 are any two points of the curve L_k, and $\sigma(t_1, t_2)$ is the length of the smallest arc of this curve with endpoints at t_1 and t_2. In ([9]), an additional requirement is imposed on L_k, ensuring the existence of a certain singular integral. With such a definition of the contour, the familiar difficulties arise that are connected with determining whether a given concrete curve possesses the indicated properties or not.

Here we consider a certain class of curves (called below class R), defined in a simpler and more natural way, and establish certain properties of curves of this class, including those possessed by the curves L_k in ([8,9]). The results obtained are used to solve equation (1) and problem (2) on a contour of class R.

1°. Let L be a simple (closed or unclosed) rectifiable curve, the coordinates of whose points, as functions of length, are given by the equations $x = x(s), y = y(s), 0 \leq s \leq l$, where l is the length of the curve L. It is known that at every point of some set E ($E \subset [0, l]$, $\operatorname{mes} E = l$) there exists a definite tangent to L. Suppose that the curve L, moreover, possesses the following two properties: 1) at each of its points there exist both left and right tangents (at each endpoint of an unclosed curve there exists one of these tangents); 2) the unit vector of the left (right) tangent at each point of the curve L is the limit from the left (from the right) of the unit tangent vectors at points of the set E. As the positive direction of the tangent we take that which coincides with the direction of increase of the arc abscissa s. The class of curves possessing the indicated properties will be called **class** R. This class includes all piecewise-smooth curves, as well as curves of bounded rotation described by I. Radon in work ([10]).

Denote by φ_s the smaller of the two nonnegative angles formed by the positive directions of the left and right tangents at the point s. The points $s \in [0, l]$ at which $\varphi_s \neq 0$ are angular points of the curve L. We shall establish the following property of curves of class R (for lack of space, here and below the proofs are not given).

Theorem 1. *Let $\beta > 0$ be any prescribed number. Then the set of angular points of a curve L of class R at which $\varphi_s \geq \beta$ is finite.*

Corollary. *The set of angular points of a curve of class R is at most countable.*

2°. Fix some number β_0 in the interval $0 < \beta < \pi/4$. On the basis of Theorem 1, a curve L of class R can be divided into a finite number of parts L_i ($i = 1, 2, \dots, n$) in such a way that at each point s of any of these parts the inequality $\varphi_s < \beta_0$ is satisfied. It can be proved that each of the parts L_i has properties analogous to those properties of smooth arcs that were noted in § 2 (1). Namely, for each of the curves L_i there exists a number R_0 such that:

1) Every circle γ with radius $\rho \leq R_0$ and with center at any point of the curve L_i cuts from this curve a unique open arc ab (a and b are the arc abscissas of the ends of the arc $a < b$).

2) Let s', s'' be any two points of the arc ab; let n' and n'' be any tangents to ab at the points s' and s'', respectively. Denote by $\theta(s', s'')$ the smaller of the two nonnegative angles formed by the positive directions of the straight lines n' and n''. Then $\theta(s', s'') < \beta_0$.

3) Let β_* be an arbitrary angle satisfying the condition $\beta_0 < \beta_* \leq \pi/2$, and let n_a, n_b be two parallel straight lines passing through a and b and forming with some tangent (left or right) at some point s on ab an obtuse angle $\beta \geq \beta_*$. Then every straight line n, parallel to the straight lines n_a and n_b and lying between them, intersects ab in exactly one point.

4) Let s_* be some fixed point of the arc ab (which may coincide with a or b); let s be a variable point of this arc; $r = r(s_*, s)$ the length of the chord s_*s. Then the function r of the variable s increases monotonically on the arc s_*b and decreases monotonically on the arc as_*. Everywhere on the arc ab the inequality $|dr| > m_0|ds|$ holds, where $m_0 = \cos\beta_0 > 0$.

Theorem 2. *For any pair of points s', s'' of a curve L of class R having no return points, the inequality $r(s', s'') > m\sigma(s', s'')$ holds, where $\sigma(s', s'')$ is the* *length of the smaller arc of the curve L with endpoints at s' and s''; $r(s', s'')$ is the length of the chord subtending it; m is some number in the interval $0 < m < 1$.*

3°. Let

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - z}\, d\tau$$

be an integral* of Cauchy type; let $\Phi^+(t)$, $\Phi^-(t)$ be its boundary values on the contour L, respectively from the left and from the right with respect to the chosen positive traversal along L; let $\varphi(t)$ satisfy a Hölder condition on L. If L belongs to class R, then from Theorem 2, on the basis of ([11]), it follows that the functions $\Phi^+(t)$, $\Phi^-(t)$ are Hölder-continuous on L, and also that the formulas

$$\Phi^+(t) = \left[1 - \frac{\alpha(t)}{2\pi}\right] \varphi(t) + \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - t}\, d\tau,$$

$$\Phi^-(t) = -\frac{\alpha(t)}{2\pi} \varphi(t) + \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - t}\, d\tau,$$

hold.

* Here and in what follows all integrals are understood in the Riemann sense.

where the integral $\int_L \frac{\varphi(\tau)}{\tau - t}\, d\tau$ is understood in the sense of the Cauchy principal value; $\alpha(t)$ is the angle of the curve L at the point t, defined as in (1), p. 436.

Since curves of class R possess the properties of the curves L_k in ([9]), all the results of that work concerning problem (2) with Hölder coefficients extend to the case of a contour L of class R. The conditions of existence, the number and the form of the solutions of problem (2) in this case remain the same as in the case of a smooth contour.

Similarly, in the case of a contour of class R, all the results of the works ([8,9]) concerning equation (1) are valid. However, it should be noted that in ([8,9]), problem (1) is considered in such formulations for which the equivalent problem of the form (2) has continuous (in the Hölder sense) coefficients. For this purpose, in ([8]) the coefficient $a(t)$ is taken in the form of a function of a certain special type, while in ([9]) by a solution of equation (1) one means a function which may fail to satisfy equality (1) at an infinite set of points of the curve L. We consider problem (1) on a contour of class R (for simplicity restricting ourselves to the case of one simple closed curve) in another, more natural formulation: we assume that $a(t)$, $b(t)$, $f(t)$ satisfy a Hölder condition on L, $a(t) \pm b(t) \neq 0$, $a(t) \neq 0$ on L, and as a solution of equation (1) we regard a function satisfying equality (1) at all points of L. Under such a formulation of the question, the solution of problem (1) acquires an essential peculiarity. It turns out that, in the general case, the equation under consideration has no continuous (in the Hölder sense) solutions. Therefore we seek the solution of equation (1) in a broader class of functions representable in the form

$$\varphi(t) = \varphi_1(t) + \xi(t), \tag{3}$$

where $\varphi_1(t)$ satisfies a Hölder condition on L, while the function $\xi(t)$ is equal to zero everywhere, except possibly at the corner points of the contour. For the latter, for any prescribed $\varepsilon > 0$, the inequality $|\xi(t)| \geq \varepsilon$ may hold only at a finite number of points.

Theorem 3. *Between the solutions of equation (1) of the form (3) and the solutions, vanishing at infinity, of the Riemann problem with boundary condition on L*

$$\Phi^+(t) = \frac{a(t) - b(t)}{a(t) + b(t)} \Phi^-(t) + \frac{f(t)}{a(t) + b(t)} \tag{4}$$

there exists a one-to-one correspondence in the sense that if (3) is a solution of equation (1), then the function

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\varphi_1(\tau)}{\tau - z} \, d\tau$$

satisfies the boundary condition (4), moreover $\Phi(\infty) = 0$; conversely, if $\Phi^\pm(z)$ is a solution of problem (4) such that $\Phi^-(\infty) = 0$, then the function

$$\varphi(t) = \varphi_1(t) + \frac{b(t)}{a(t)} \left[1 - \frac{\alpha(t)}{\pi} \right] \varphi_1(t), \tag{5}$$

where $\varphi_1(t) = \Phi^+(t) - \Phi^-(t)$, is a solution of equation (1) having the form (3).

From (5) it is clear that the solution of equation (1) is discontinuous on L in the case when, at each corner point t_k of the contour, at least one of the equalities $b(t_k) = 0$ or $\Phi^+(t_k) - \Phi^-(t_k) = 0$ is not satisfied. If the index $\varkappa$ of the boundary-value problem (4) is positive, then the arbitrary constants entering the solution of equation (1) can be chosen so that at $\varkappa$ prescribed corner points this solution is continuous.

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Note: Figure translations are in progress. See original paper for figures.