A NEW METHOD FOR SOLVING THE CAUCHY PROBLEM GLOBALLY FOR QUASILINEAR EQUATIONS

1961

SovietRxiv

Abstract

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MATHEMATICS

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A NEW METHOD FOR SOLVING THE CAUCHY PROBLEM GLOBALLY FOR QUASI-LINEAR EQUATIONS

(Presented by Academician M. V. Keldysh on 10 XII 1960)

For the construction of a smooth (classical) solution of the Cauchy problem for a system of quasilinear equations of hyperbolic type

$$\frac{du_i}{dt} + \frac{\partial \varphi_i(u, t, x)}{\partial x} = f_i(u, t, x), \tag{1}$$

$$u_i(0, x) = u_i^0(x) \qquad (i = 1, \dots, n)$$

regular methods have been developed, chiefly the method of characteristics, which make it possible to construct the solution up to the formation of singularities in it (discontinuity of the solution, unboundedness of derivatives, etc.). However, for constructing generalized (discontinuous) solutions of problem (1), these methods in their direct form prove to be inapplicable.

On the basis of a new definition of stability of a generalized solution of problem (1), we propose a method for constructing discontinuous solutions, using as its elements the methods for constructing classical solutions. This method is also convenient for the numerical solution of problem (1).

1. The concept of a generalized solution of problem (1) and its stability. Let the function $\Phi(t, x) = \{\Phi_1(t, x); \dots; \Phi_n(t, x)\}$ be defined and continuous for $0 \le t \le T$, $x_1(t) \le x \le x_2(t)$, and almost everywhere in this domain have bounded first derivatives satisfying the equations

$$\frac{\partial \Phi_i(t, x)}{\partial t} + \varphi_i\left(\frac{\partial \Phi(t, x)}{\partial x}, t, x\right) = \int_a^x f_i\left(\frac{\partial \Phi(t, \xi)}{\partial \xi}, t, \xi\right) d\xi \qquad (i = 1, \dots, n) \tag{2}$$

and the initial condition

$$\Phi_i(0, x) = \Phi_i^0(x) = \int_a^x u_i^0(\xi)\, d\xi \qquad (i = 1, \dots, n). \tag{3}$$

We shall call the function $u(t,x) = \{\partial\Phi_1(t,x)/\partial x; \dots; \partial\Phi_n(t,x)/\partial x\} = \partial\Phi(t,x)/\partial x$ a generalized solution of the Cauchy problem (1).

We shall say that a set $\mathcal{E}$ of generalized solutions $u(t,x)$ forms a stability class of solutions of the system of equations (1), if:

a) For every bounded measurable function $u^0(x)$ there exists $u(t,x) \in \mathcal{E}$ such that

$$\int_a^x [u_i(0,\xi) - u_i^0(\xi)]\, d\xi \equiv 0 \qquad (i = 1, \dots, n).$$

b) For any $u(t,x) \in \mathcal{E}$, $\varepsilon > 0$, and $0 \leq \tau < t$, there exists $\delta > 0$ such that

$$\left| \int_a^x [u_i(t,\xi) - u_i(t,\xi)]\, d\xi \right| \leq \varepsilon \qquad (i = 1, \dots, n) \tag{4}$$

for all $\bar{u}(t,x) \in \mathcal{E}$ and satisfying the condition:

$$\left| \int_a^z [u_i(\tau,\xi) - \bar{u}_i(\tau,\xi)]\, d\xi \right| \leqslant \delta \qquad (i = 1, \dots, n) \tag{5}$$

$$x_1(\tau) \leqslant z \leqslant x_2(\tau).$$

c) Every smooth (classical) solution of system (1) belongs to $\mathcal{E}$. We shall call a generalized solution $u(t,x)$ of the system of equations (1) stable if there exists a class $\mathcal{E}$ containing $u(t,x)$.

This definition of stability of discontinuous solutions reflects the basic facts known in the theory of systems of quasilinear equations of hyperbolic type, and also establishes a metric in the space of generalized solutions. Let us note that this definition is automatically satisfied by generalized (in the usual sense) solutions of systems of linear equations.

Conditions (4), (5) on the class $\mathcal{E}$ give a criterion of stability in place of various criteria of stability of an individual generalized solution (“shore conditions,” “convex-envelope conditions,” “herringbone conditions,” etc.), valid for narrow classes of systems (1).

As is known ([1,2]), problem (1) is equivalent to the Cauchy problem (2), (3) for the potential of the generalized solution $\Phi(t,x)$. Our definition of stability of discontinuous solutions of problem (1) requires strong stability (in the C-norm) of solutions of the Cauchy problem (2), (3). From the definition of stability there follows the uniqueness of a stable generalized solution of problem (1) in the class $\mathcal{E}$. The uniqueness theorem for the solution of the Cauchy problem reduces to proving the uniqueness of the stability class of solutions of the system of equations (1).

2. Method for constructing generalized solutions. The method of characteristics makes it possible to construct the solution $u(t,x)$ of problem (1) with a smooth function $u^0(x)$ up to the moment when a singularity arises in the solution. Therefore we shall assume that the function $u^0(x)$ has singularities (discontinuities of the function, unboundedness of the first and higher derivatives, etc.). We shall suppose that at $x = 0$ $u^0(x)$ has an isolated singularity, and we shall consider the solution of problem (1) in the domain of influence of the interval $|x| \leqslant b$, where b is sufficiently small.

Let us smooth the initial function $u^0(x)$ on the interval $|x| \leqslant \varepsilon$ so that the smoothed function $\tilde{u}^0_\varepsilon(x)$ has no singularities for $|x| \leqslant b$ and coincides with $u^0(x)$ outside the interval $|x| \leqslant \varepsilon$; in doing so, let us satisfy the conditions

$$\int_{-\varepsilon}^{\varepsilon} [\tilde{u}^0_{i\varepsilon}(\xi) - u^0_i(\xi)]\, d\xi = 0 \qquad (i = 1, \dots, n), \tag{6}$$

which, when applied to problem (5), gives

$$\widetilde{\Phi}^0_\varepsilon(x) \equiv \Phi^0(x) \quad \text{for } |x| \geqslant \varepsilon. \tag{7}$$

Such a smoothing can evidently always be carried out, and in many ways. If the smoothing is performed with the aid of a bounded function $\tilde{u}^0_\varepsilon(x)$: $|\tilde{u}^0_{i\varepsilon}(x)| < M$ and $|u^0_i(x)| < M$, then

$$|\widetilde{\Phi}^0_{i\varepsilon}(x) - \Phi^0_i(x)| < M\varepsilon \qquad (i = 1, \dots, n). \tag{8}$$

Inequalities (8) and (4), (5) show that the solution $\tilde{u}_\varepsilon(t,x)$ of the Cauchy problem

$$\tilde{u}_\varepsilon(0,x) = \tilde{u}^\varepsilon_0(x) \tag{9}$$

for system (1) must be close to the solution $u(t,x)$ of problem (1).

We shall solve problem (9) for system (1). Either we determine the solution of this problem for $0 \leqslant t \leqslant T$, or else at $t = t_1(\varepsilon) < T$ singularities will again arise in the solution. Smoothing the function $u_\varepsilon(t_1(\varepsilon),x)$ on intervals of length 2ε containing the singularities of the function $\tilde{u}_\varepsilon(t_1(\varepsilon),x)$, by means of the bounded function $\tilde{u}^1_\varepsilon(x)$, and again satisfying conditions (6), we solve for system (1) the Cauchy problem $\tilde{u}_\varepsilon(t_1(\varepsilon),x) = \tilde{u}^1_\varepsilon(x)$. Continuing this process, we shall, in general, define the functions $\tilde{u}_\varepsilon(t,x)$ and $\tilde{\Phi}_\varepsilon(t,x)$ for $0 \leqslant t \leqslant T$. These functions are continuous everywhere, except for the smoothing intervals, outside which $\tilde{\Phi}_\varepsilon(t,x)$ is differentiable.

If the limit $\lim_{\varepsilon\to 0} \tilde{\Phi}_\varepsilon(t,x) = \Phi(t,x)$ exists, then $u(t,x) = \partial\Phi(t,x)/\partial x$ will be a generalized and stable solution of the Cauchy problem (1), since we assume nothing about the character of the smoothing except condition (6). This condition ensures the "potentiality" of the smoothing or, in other words, the fulfillment of the "conservation laws" of the system of equations (1). It is easy to give examples in which abandoning condition (6) leads to incorrect results.

3. Application of the method of "potential smoothing" to the construction of discontinuous solutions of a single quasilinear equation. We shall consider the Cauchy problem

$$\frac{\partial u}{\partial t}+\frac{\partial \varphi(u,t,x)}{\partial x}=f(u,t,x), \qquad u(0,x)=u_0(x) \tag{10}$$

in the domain of influence of the initial data on the segment $x_1 \leqslant x \leqslant x_2$ of the initial axis, assuming that $\varphi(u,t,x)$, $f(u,t,x)$ are twice continuously differentiable, and $|u_0(x)|<M$.

Theorem. *There exists a unique stability class $\mathcal{E}$ of generalized solutions of the Cauchy problem under consideration. Each of these solutions can be constructed by the method of "potential smoothing" of singularities, i.e. for each solution its potential $\Phi(t,x)=\lim_{\varepsilon\to 0}\widetilde{\Phi}_\varepsilon(t,x)$.*

Without giving the proof here, we illustrate its main idea in the simplest case. Namely, let us suppose that $f(u,t,x)\equiv 0$ [(2)], and that the function $u_0(x)$ is piecewise continuous and, in a neighborhood $|x|<b$ of the point $x=0$, has at $x=0$ an isolated singularity such that the limit of the quotient

$$\lim_{\Delta x\to 0}\frac{\varphi'_u(u_0(x+\Delta x),0,0)-\varphi'_u(u_0(x),0,0)}{\Delta x}=A(x) \tag{11}$$

does not exist at $x=0$, and $A(x)<0$ for $|x|\leqslant b$ $(x\neq 0)$. Without loss of generality, we shall assume that the characteristics of problem (10) $x=X(t,x_0^{(1)},u_0(x_0^{(1)}))$, $x=X(t,x_0^{(2)},u_0(x_0^{(2)}))$, $x=X(t,x_0^{(3)},u_0(x_0^{(3)}))$, where $x_0^{(1)}\cdot x_0^{(2)}>0$, and $x_0^{(1)}\cdot x_0^{(3)}<0$, satisfy the requirement: the first two intersect each other later (for larger t) than one of them intersects the third characteristic. This assumption means that in the neighborhood under consideration there arises only one shock wave emerging from the point $x=0$. We now apply the indicated method.

Obviously, from the boundaries of the smoothing intervals there emerge the true characteristics of equation (10) and of the Cauchy problem $u(0,x)=u_0(x)$. The functions $\tilde{u}_\varepsilon(t,x)$, $\widetilde{\Phi}_\varepsilon(t,x)$ will be continuous everywhere, except for the smoothing intervals of the straight lines $t=t_k(\varepsilon)$. Figure 1 depicts these intervals, as well as the intervals of the true characteristics of problem (10), lying between the smoothing intervals, for two values $\varepsilon=\varepsilon_1$, and $\varepsilon=\varepsilon_2<\varepsilon_1$. The curvilinear trapezoids corresponding to the case $\varepsilon=\varepsilon_1$ are shown by a solid line, and those corresponding to the case $\varepsilon=\varepsilon_2$ by a dashed line. Thus, the functions $\tilde{u}_\varepsilon(t,x)$, $\widetilde{\Phi}_\varepsilon(t,x)$ are continuous and differentiable outside the corresponding "shock-wave belt" made up of curvilinear trapezoids. By the assumptions made, the "belt"

Fig. 1

Figure 1: Fig. 1

corresponding to ε_2 lies entirely inside the "belt" corresponding to the value ε_1. Outside these "belts" ,

Fig. 1

$$\tilde{u}_{\varepsilon_1}(t,x) \equiv \tilde{u}_{\varepsilon_2}(t,x).$$

Since

$$|\tilde{u}_{\varepsilon_1}(t,x)| < M, \qquad |\tilde{u}_{\varepsilon_2}(t,x)| < M,$$

it follows that

$$|\widetilde{\Phi}_{\varepsilon_1}(t,x) - \widetilde{\Phi}_{\varepsilon_2}(t,x)| < 2M\varepsilon.$$

Thus, the sequence $\widetilde{\Phi}_{\varepsilon}(t,x)$ is fundamental and converges uniformly. Its limit

$$\Phi(t,x) = \lim_{\varepsilon \to 0} \widetilde{\Phi}_{\varepsilon}(t,x)$$

is a continuous function, which is the potential of the generalized stable solution.

The indicated method of proof is then generalized to the case of arbitrary bounded measurable functions $u_0(x)$.

The totality of the solutions constructed forms, as is easy to see, the class $\mathcal{E}$; here $\delta(t,\varepsilon) = \varepsilon$. The uniqueness of the class $\mathcal{E}$ is easily established.

4. The indicated method may be regarded as a possible method for the numerical solution of the Cauchy problem (1) and, in particular, may be applied to the construction of discontinuous solutions of the equations of hydrodynamics.

The author expresses his gratitude to Corresponding Member of the Academy of Sciences of the USSR A. N. Tikhonov for discussion of the question and valuable comments.

Received
9 XII 1960

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.