Reports of the Academy of Sciences of the USSR

1961

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1961. Vol. 141, No. 1

MATHEMATICS

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STABILITY IN THE CASE OF NEUTRALITY OF THE LINEAR APPROXIMATION

(Presented by Academician M. V. Keldysh, 2 VI 1961)

As is known, linearization is the basic method for studying the stability of a singular point of a system of ordinary differential equations. The linearized system is divided, generally speaking, into three components: stable, neutral, and unstable. If there is an unstable component, then the question of stability is decided negatively, independently of the other components. In the presence of only one stable component the original system proves to be stable. The proof of this assertion constitutes the subject of Lyapunov' s stability theory. The question remains open if, in addition to a stable component, there is also a neutral component. The purpose of the present note is to establish a stability criterion precisely for this latter case.

Without loss of generality one may assume that there is only a neutral component, since the general case is reduced to this simpler one by the following uncomplicated device. In the original system set equal to zero all variables belonging to the stable component. This gives a "truncated" system having only the neutral component. It can be shown that if the "truncated" system proves stable, then the original system is also stable. The principal difficulty in studying systems with a neutral component consists in the fact that the motion near a singular point turns out, in the principal term, to be almost periodic, while stability or instability appears only in the subsequent approximations.

Therefore, in order to clarify the question of stability, it is necessary to carry out a separation of the motion ([1]) and to eliminate the principal motion. Let the system under study have the form

$$\frac{du}{dt} = U(u), \tag{1}$$

and suppose that $u = u_0 = \text{const}$ is a solution of system (1). Moving the origin of coordinates to the point u_0, $u = u_0 + \varepsilon x$, let us consider small values of the parameter ε, which corresponds to a small neighborhood of u_0 in the original variables. System (1) in the new variables has the form

$$\frac{dx}{dt} = A(x) + \varepsilon A_1(x) + \frac{\varepsilon^2}{2!} A_2(x) + \cdots \tag{2}$$

Here $A, A_1, A_2, \ldots$ are homogeneous polynomials of respectively the first, second, third, etc. degrees. According to the basic assumption, the matrix A has purely imaginary eigenvalues. We apply the scheme of separation of motions to system (2). This means that it is necessary to introduce a change of variables

$$y = x + \varepsilon Q_1(x) + \frac{\varepsilon^2}{2!} Q_2(x) + \cdots \tag{3}$$

so that the new system

$$\frac{dy}{dt} = A(y) + \varepsilon B_1(y) + \frac{\varepsilon^2}{2!} B_2(y) + \cdots \tag{4}$$

would admit a separation of motions. In (1) it is shown that the functions Q_n and B_n are computed by the formulas

$$B_n(y) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\frac{\partial f}{\partial y}\right)^{-1} C_n(f)\, dt, \tag{5}$$

$$Q_n(y) = \lim_{T\to\infty} \frac{1}{T} \int_0^T (T-t) \left(\frac{\partial f}{\partial y}\right)^{-1} \left[C_n(f) - B_n(f)\right] dt. \tag{6}$$

In these formulas $f = f(y,t)$ is the solution of the unperturbed equation $\partial f/\partial t = A(f)$, satisfying the initial condition $f(y,0) = y$. The functions C_n are computed from the already known B_{n-1}, Q_{n-1}. For example, $C_1(x) = A_1(x)$,

$$C_2(x) = A_2(x) + 2\left[\frac{dQ_1}{dx} A_1 - \frac{dB_1}{dx} Q_1\right] - \frac{d^2 A}{dx^2} Q_1 Q_1,$$

and so on. It is not difficult to verify that, in our case, when A_n is a homogeneous polynomial of degree $(n+1)$, the functions C_n, B_n, Q_n are also homogeneous polynomials in their arguments of degree $(n+1)$.

The functions $B_n(y)$ are of principal interest to us, since it is precisely they that determine the stability of the system. These functions take the simplest form when the matrix A is diagonal. Since the matrix A, according to the condition, has purely imaginary eigenvalues, the variables and the coefficients of the equations must also be regarded as complex numbers. The coefficients are, of course, not arbitrary complex numbers, for the system has been obtained from a real system. If by x^{α^*} we denote the variable complex conjugate to x^α, then it is easy to see that the coefficients satisfy conditions of the type $A^{\alpha^*}_{\beta^*\gamma^*} = A^\alpha_{\beta\gamma}$. For the coefficients of the polynomials B, C, and Q, analogous equalities hold.

Let us denote by $i\omega_\alpha$ the eigenvalues of the matrix A and note that

$$\omega_{\alpha^*} = -\omega_\alpha. \tag{7}$$

For a diagonal matrix, the calculations in formula (5) are not hard to carry through to the end. Indeed, in this case the solution $f(y,t)$ has the form $f^\alpha = y^\alpha \exp(i\omega_\alpha t)$. Substituting into (5), we obtain that $B^\alpha_{\beta\gamma} y^\beta y^\gamma = C^\alpha_{\beta\gamma} y^\beta y^\gamma \times$

$$\times \lim_{T\to\infty} \int_0^T e^{it(\omega_\beta+\omega_\gamma-\omega_\alpha)}\, dt.$$

But the resulting limit is different from zero (and equal to unity) only in the case when the equality

$$\omega_\alpha - (\omega_\beta + \omega_\gamma) = 0, \tag{8}$$

is satisfied, which it is natural to call the **condition of internal resonance of the second order**.

In exactly the same way, in computing B_2, only those terms remain for which there is a **third-order resonance**

$$\omega_\alpha - (\omega_\beta + \omega_\gamma + \omega_\delta) = 0. \tag{9}$$

Let us note that in any system there is necessarily a third-order resonance. Indeed, putting $\beta = \alpha,\ \delta = \gamma^*$, we see that (9) follows from (7). The situation is different for second-order resonance—as a rule, it is absent. More precisely, this important conclusion may be formulated as follows.

Consider a system containing a parameter. In this case the basic frequencies of the system ω_α, and consequently also the combination frequencies $\omega_{\alpha\beta\gamma}, \omega_{\alpha\beta\gamma\delta}$ (the left-hand sides of equalities (8) and (9)) will be functions of this parameter. Some of the combination frequencies, namely $\omega_{\alpha\alpha\gamma\gamma^*}$, are equal to zero for all values

of the parameter. This is an identical resonance of third order. The others, in particular $\omega_{\alpha\beta\gamma}$, the frequencies of second order, vanish only at isolated points, determining critical values of the parameter. The very interesting question of the passage of a system through such a critical state (in particular, through a zero root of the matrix A, when (8) follows from (7)) is not considered here. The present note is devoted to the analysis of the general case in which the system has no resonances except the identical one. Then the matter reduces to the study of a system of equations of the form (summation is performed only over the index α):

$$\frac{d\eta^k}{dt} = -\eta^k \left(E_{k\alpha}\eta^\alpha\right), \tag{10}$$

where $E_{\alpha\beta} = -\frac{\varepsilon^2}{2!}\left(B^{\alpha}_{\alpha\beta*} + B^{\alpha*}_{\alpha*\beta*}\right)$. This system is obtained from (4) if, instead of y^k, one introduces new variables η^k by the formula $\eta^k = |y^k|^2 = y^k y^{k*}$, and it has half as many equations as (4). Thus the question of the stability of the system (4) is reduced to the question of the stability of the system (10) in the cone $\eta^k \geq 0$.

Let us note in this connection one essential property of the system (10). If some variable η^k is equal to zero at $t = t_0$, then it is identically equal to zero, as follows easily from the form of the system (10). Therefore, in particular, every variable preserves its sign for all values of t. If all variables except one are set equal to zero, then it is not difficult to obtain the necessary condition for stability

$$E_{kk} > 0. \tag{11}$$

Positive definiteness of the symmetric part of the matrix $E_{\alpha\beta}$,

$$\left(\left(\frac{E_{\alpha\beta} + E_{\beta\alpha}}{2}\right)\right) > 0 \tag{12}$$

is the simplest sufficient condition for stability. This assertion is easily verified by adding all the equations of the system (10).

Let us proceed to the establishment of necessary and sufficient conditions. An important role in the formulation of the stability criterion, as also for linear equations, is played by the invariant rays of the system, i.e. solutions of the form $\eta^k = \eta_0^k \eta(t)$. Substituting these expressions in (10), we have

$$\frac{d\eta}{dt} = -E\eta^2, \qquad \eta(0) = 1, \tag{13}$$

$$\eta_0^k \left[E_{k\alpha}\eta_0^\alpha - E\right] = 0. \tag{14}$$

Here E is a parameter analogous to an eigenvalue in linear systems. However, the magnitude of this parameter, in view of its proportionality to the length of the vector of initial data η_0^α, has no significance. What is important is the sign of E, which, as is seen from equation (13), determines stability.

The method for finding all invariant rays of the system (10) is suggested by the form of the system (14). First leave in each of the equations only the second factor. The resulting basic system of linear equations is

$$E_{k\alpha}\eta_0^\alpha = E. \tag{15}$$

If the matrix $E_{\alpha\beta}$ is nonsingular, then for every value of the parameter E the system (15) has a unique solution. These solutions fill an invariant straight line consisting of two rays–stable ($E > 0$) and unstable ($E < 0$). If, however, the matrix is singular, then a solution exists only for $E = 0$, and then it is determined up to proportionality. Again we have an invariant straight line, this time neutral. All solutions of the nonlinear system (14) can be obtained by leaving in each of the equations either the first or the second factor. In all, therefore, 2^n solutions are obtained, including the one considered above and the identical one: $\eta^k = 0$. It is easy

one sees that this procedure corresponds to the independent study of system (10) on all possible faces of the cone $\eta^k \geqslant 0$. The concept of an invariant ray makes it possible to formulate the

Stability criterion. *In order that system (10) be stable in the cone $\eta^k \geqslant 0$, it is necessary and sufficient that there be not a single neutral or unstable ray either inside or on the boundary of the cone.*

The necessity is obvious.

The sufficiency is proved, just as in linear systems, by constructing a Lyapunov function. We indicate the guiding idea of this construction. Rewrite system (10) in the form $d\ln\eta^k/dt = -E_{k\alpha}\eta^\alpha$. Now multiply both sides of each of the equations by z_k and add all n equations. We obtain the relation

$$\frac{d\ln\Phi}{dt} = -\Psi. \tag{16}$$

Here the notation introduced is:

$$\ln\Phi = \sum_k z_k \ln\eta^k, \qquad \Psi = \sum_\alpha \zeta_\alpha\eta^\alpha, \qquad \zeta_\alpha = \sum_k E_{k\alpha}z^k.$$

It can be shown that, when the conditions of the criterion are fulfilled, there exist positive z^k, to which positive ζ_α correspond. The resulting function Φ cannot serve as a Lyapunov function only because it vanishes on the faces of the positive cone. This defect can be corrected by adding to Φ (with a sufficiently small weight, so as not to spoil the negative definiteness of the derivative) the Lyapunov functions of the faces. The existence of Lyapunov functions of smaller dimensions we suppose proved (by induction).

In conclusion let us analyze ([2]) systems of four equations (two degrees of freedom in the theory of oscillations). Equations (10) in this case admit integration in quadratures. By changing scales one can arrange that $E_{11} = 1$, $E_{22} = 1$. Therefore such systems form a two-parameter family, and they may be represented by points in the plane (α, β), where $\alpha = E_{12}$, $\beta = E_{21}$.

It is not difficult to verify that unstable systems lie below the negative branch of the hyperbola $\alpha\beta - 1 = 0$. Above the straight line $\alpha + \beta + 2 = 0$ lies the region of monotone stability. Between these lines are situated formally stable systems, whose solutions, however, may increase before beginning to decay. The degree of "swing-up" in such systems (i.e., the possible initial growth of the amplitude of oscillations) is roughly estimated by the number $|\alpha| + |\beta|$. It is clear that, with large "swing-up," such systems may prove to be practically unstable.

For systems with two frequencies it is not difficult to construct directly a Lyapunov function linear in the variables η_1, η_2. Therefore they, along with single-frequency systems, may serve as the starting point of an induction proof for general systems.

Received23 V 1961

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Note: Figure translations are in progress. See original paper for figures.

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