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Abstract

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AN EXAMPLE OF NON-UNIQUENESS OF A GENERALIZED SOLUTION OF A QUASILINEAR SYSTEM OF EQUATIONS

(Presented by Academician M. V. Keldysh, 15 IX 1960)

To define the concept of a generalized solution of quasilinear hyperbolic systems, one has to introduce "viscous terms" tending to zero ([1]). In papers ([2], [3]) it was shown that the form of the generalized solution may depend on these viscous terms.

Below we give an example of a hyperbolic system of two quasilinear equations for which different generalized solutions of the same Cauchy problem are obtained from solutions of parabolic systems with identical viscous terms. The system has the form proposed by S. K. Godunov. The example arose as a result of a discussion with S. K. Godunov of questions connected with his paper ([2]). For clarity we shall repeat some of the assertions given there.

Consider the Cauchy problem for the system

$$\frac{\partial u}{\partial t} + \frac{\partial L_u(u,v)}{\partial x} = 0, \qquad \frac{\partial v}{\partial t} + \frac{\partial L_v(u,v)}{\partial x} = 0 \tag{1}$$

with initial conditions $u_0(x), v_0(x)$, constant for $x > 0$ and $x < 0$,

$$\begin{aligned} u_0(x) &= u_1, \qquad v_0(x) = v_1, \qquad x > 0; \\ u_0(x) &= u_2, \qquad v_0(x) = v_2, \qquad x < 0. \end{aligned} \tag{2}$$

We shall seek the generalized solution of problem (1), (2) as the limit, as $\varepsilon \to 0$, of solutions of the system

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial L_u(u_\varepsilon, v_\varepsilon)}{\partial x} = \varepsilon \frac{\partial^2 u_\varepsilon}{\partial x^2},$$

$$\frac{\partial v_\varepsilon}{\partial t} + \frac{\partial L_v(u_\varepsilon, v_\varepsilon)}{\partial x} = \varepsilon \frac{\partial^2 v_\varepsilon}{\partial x^2} \tag{3}$$

with initial conditions

$$u_\varepsilon(x,0)=u_0^\varepsilon(x),\qquad v_\varepsilon(x,0)=v_0^\varepsilon(x);\tag{4}$$

$$u_0^\varepsilon(\pm\infty)=u_{1,2},\qquad v_0^\varepsilon(\pm\infty)=v_{1,2};$$

$$u_0^\varepsilon\to u_0,\qquad v_0^\varepsilon\to v_0\quad\text{as }\varepsilon\to 0.$$

Let $L_u(u_1,v_1)=L_u(u_2,v_2)$, $L_v(u_1,v_1)=L_v(u_2,v_2)$. We shall find stationary solutions of problem (3), (4) of the form $u_\varepsilon=u(\xi)$, $v_\varepsilon=v(\xi)$, $\xi=x/\varepsilon$. For these solutions

$$\frac{dL_u(u,v)}{d\xi}=\frac{d^2u}{d\xi^2},\qquad \frac{dL_v(u,v)}{d\xi}=\frac{d^2v}{d\xi^2},$$

and, therefore,

$$Lu+c_1-\frac{du}{d\xi}=0,\qquad L_v+c_2-\frac{dv}{d\xi}=0.$$

In order that, as $\xi\to\pm\infty$, the functions u,v satisfy the required conditions, the points $A_1=(u_1,v_1)$, $A_2=(u_2,v_2)$ must be stationary points of the function

$$\Lambda=L+c_1u+c_2v.$$

It is easy to see that, in the (u,v)-plane, the solution of our problem is a trajectory joining the points A_1 and A_2 and going orthogonally to the level lines of the function Λ; along the trajectory Λ increases with increasing ξ.

Let $u_1=5,\ u_2=v_1=v_2=0$,

$$L(u,v)=(u^2-2u+v^2+1)(u^2-6u+v^2-3)+25u,$$

$$\Lambda(u,v)=(u^2-2u+v^2+1)(u^2-6u+v^2-3).$$

The pattern of the level lines of the function Λ is shown in Fig. 1. The points A_1,A_2 will be stationary points of the function Λ. There are two trajectories, symmetric with respect to the u-axis, going from A_1 to A_2 (we note that the segment of the u-axis between these points is not such a trajectory). It is clear that, as $\varepsilon\to 0$, the functions $u_\varepsilon,v_\varepsilon$ corresponding to these trajectories converge to the same functions $u(x,t)=u_0(x)$, $v(x,t)=v_0(x)$.

Fig. 1

Fig. 1

Figure 1: Fig. 1

Let us now note that the problem (3), (4) is satisfied by functions for which $v_\varepsilon(x,0) = 0$, $v_\varepsilon = 0$,

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial L_u(u_\varepsilon, 0)}{\partial x} = \varepsilon \frac{\partial^2 u_\varepsilon}{\partial x^2}. \tag{5}$$

As shown in (4), for initial data $u_0^\varepsilon(x) \to u_0(x)$ (in particular, for $u_0^\varepsilon(x) = u_0(x)$) the solutions of equation (5) converge to a function $u(x,t) = u(x/t)$, not coinciding with the function $u_0(x)$.

Both pairs of limiting functions are generalized solutions of the quasilinear system (1) and satisfy condition (2).

Whether system (3) has a unique solution of the Cauchy problem for discontinuous initial data of the form (2), we do not know.

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CITED LITERATURE

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2. S. K. Godunov, *DAN*, **134**, No. 6 (1960).

3. V. F. Dyachenko, *DAN*, **136**, No. 1 (1961).

4. A. S. Kalashnikov, *DAN*, **127**, No. 1 (1959).

Note: Figure translations are in progress. See original paper for figures.