Reports of the Academy of Sciences of the USSR

1961

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1961. Vol. 138, No. 5

GEOPHYSICS

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ON THE INTEGRATION OF THE EQUATIONS OF THE THEORY OF SEA CURRENTS IN MULTIPLY CONNECTED DOMAINS

(Presented by Academician V. V. Shuleikin, 28 XI 1960)

Let us first consider the basic equations of Ekman' s theory (^{1}) for steady wind-driven currents in a sea of homogeneous density. The general problem in this case (including also the determination of current velocities at individual levels) can be reduced to solving the following system of equations for the unknowns $S_\lambda(\lambda, \theta)$, $S_\theta(\lambda, \theta)$, $\zeta(\lambda, \theta)$:

$$\frac{\partial \zeta}{a_0 \sin\theta\, \partial\lambda} = A(\lambda, \theta) S_\lambda + B(\lambda, \theta) S_\theta + F_\lambda(\lambda, \theta); \qquad (1)$$

$$\frac{\partial \zeta}{a_0 \partial\theta} = -B(\lambda, \theta) S_\lambda + A(\lambda, \theta) S_\theta + F_\theta(\lambda, \theta); \qquad (2)$$

$$\frac{\partial S_\lambda}{\partial \lambda} + \frac{\partial}{\partial \theta}(\sin\theta \cdot S_\theta) = 0. \qquad (3)$$

Equations (1)—(3) are written in a spherical coordinate system: λ is measured along a circle of latitude in the easterly direction, θ along a meridian from the South Pole; a_0 is the radius of the Earth; $\zeta(\lambda, \theta)$ is the sea level; S_λ and S_θ are the components of the total transport;

$$S_\lambda = \int_0^H u\, dz, \qquad S_\theta = \int_0^H v\, dz,$$

where u is the zonal and v the meridional component of velocity; $H(\lambda, \theta)$ is the variable depth of the sea. The functions $A, B, F_\lambda, F_\theta$ are known; $A(\lambda, \theta) > 0$; F_λ and F_θ depend on the known components of the wind stress.

In Ekman' s work (^{1}), to solve the problem, a single equation was formed for $\zeta(\lambda, \theta)$ with very complicated boundary conditions. As A. I. Felzenbaum (^{2}) showed, the discussion of the problem can be substantially simplified if one forms an equation for the function of total transports $\psi(\lambda, \theta)$, whose determination in a simply connected domain reduces to solving the Dirichlet problem for a certain elliptic equation.

However, consideration of only simply connected domains in the study of sea currents is clearly insufficient; in many problems we encounter multiply connected domains. Such problems have their own specific features, and methods for solving them have not yet been developed. The following method can be proposed for solving the problem for a multiply connected domain.

Let D be a multiply connected domain in which the solution of equations (1)–(3) is sought; Γ_0 is the outer contour; $\Gamma_1, \ldots, \Gamma_n$ are the inner contours; t is the direction of the tangent to the contour; ν is the direction of the inner normal. Traversal of the contour is positive if the domain D remains on the left. With the adopted notation, the boundary conditions of the problem can be written in the form:

$$S_\nu\big|_{\Gamma_k} = 0, \qquad k = 0, 1, \ldots, n. \tag{4}$$

Thus, we must solve equations (1)–(3) in the multiply connected domain D under the boundary conditions (4). We note that the level ζ is sought to within an arbitrary constant.

By virtue of equation (3), one can introduce the function of total transports by the following relations:

$$S_\lambda = \frac{\partial\psi}{a_0 \partial\theta}, \qquad S_\theta = -\frac{\partial\psi}{a_0 \sin\theta\, \partial\lambda}. \tag{5}$$

Substituting (5) into (1) and (2) and eliminating ζ from these equations by cross differentiation, we arrive at the equation for $\psi(\lambda, \theta)$:

$$\frac{1}{\sin\theta}\frac{\partial}{a_0\partial\theta}\left(A\sin\theta\,\frac{\partial\psi}{a_0\partial\theta}\right) + \frac{1}{a_0\sin\theta}\frac{\partial}{\partial\lambda}\left(A\frac{\partial\psi}{a_0\sin\theta\,\partial\lambda}\right) + \frac{\partial\psi}{a_0\partial\theta}\frac{\partial B}{a_0\sin\theta\,\partial\lambda} - \frac{\partial\psi}{a_0\sin\theta\,\partial\lambda}\frac{\partial B}{a_0\partial\theta} = \frac{1}{\sin\theta}\frac{\partial}{a_0\partial\theta}(\sin\theta\cdot F \tag{6}$$

The boundary conditions for equation (6) follow from conditions (4). The function ψ must be constant on each boundary contour. Since the function ψ is determined up to an additive constant, one may set

$$\psi|_{\Gamma_0} = 0. \tag{7}$$

Then we have

$$\psi|_{\Gamma_k} = Q_k, \qquad k = 1, \dots, n. \tag{8}$$

The constants $Q_1, \dots, Q_n$ are unknown to us. Differences of the quantities Q_k determine the total water discharges in the corresponding straits. It is therefore clear that the quantities $Q_1, \dots, Q_n$ cannot be assigned arbitrarily, but must be determined from the solution of the problem.

A. I. Felzenbaum drew the author' s attention to one particular case in which these constants can be found without using equation (6). If one considers the domain D in the form of a ring $0 \leq \lambda \leq 2\pi$, $\theta_1 \leq \theta \leq \theta_2$, then for constant depth and a zonal wind field it is possible to find a solution of equations (1)–(3) in the form $S_\theta \equiv 0$, $S_\lambda = S_\lambda(\theta)$, $\zeta = \zeta(\theta)$. It is evident that conditions (4) are satisfied. The discharge Q is found as:

$$Q = \int_{\theta_1}^{\theta_2} S_\lambda(\theta)\, a_0\, d\theta.$$

Let us consider the problem in general form. Suppose that we have succeeded in finding the function ψ. Then from relations (5) we find S_λ and S_θ. Substituting them into equations (1) and (2), we arrive at the problem of determining ζ from its known partial derivatives. Equation (6), which the function ψ must satisfy, expresses the condition of equality of the mixed derivatives of the function ζ. But, since we seek a single-valued function ζ in the multiply connected domain D, this condition alone is not sufficient. When equation (6) is satisfied, the necessary and sufficient conditions for the possibility of determining ζ in the domain D are the following conditions:

$$\oint_{\Gamma_k} \frac{\partial \zeta}{\partial \lambda}\, d\lambda + \frac{\partial \zeta}{\partial \theta}\, d\theta = 0, \qquad k = 1, \dots, n. \tag{9}$$

Using expressions (1), (2), (4), (5), conditions (9) can be written in the form

$$\oint_{\Gamma_k} A \frac{\partial \psi}{\partial \nu}\, d\sigma_t + \oint_{\Gamma_k} F_t\, d\sigma_t = 0, \qquad k = 1, \dots, n. \tag{10}$$

Here $d\sigma_t$ is an element of arc of the contour Γ_k; F_t is the component of the vector $\mathbf{F} = \{F_\lambda, F_\theta\}$ tangent to the contour. Thus, the sought function of total transports must satisfy equation (6) and the boundary conditions (7), (8), (10). We shall show that such a function can be found uniquely.

Indeed, it is easy to show that the solution of equation (6) under conditions (7) and (8) can be represented in the form

$$\psi = \psi_0 + \sum_1^n Q_k \psi_k. \tag{11}$$

The function ψ_0 is a solution of the inhomogeneous equation (6) with zero values on all contours $\Gamma_0, \Gamma_1, \dots, \Gamma_n$. The functions $\psi_1, \dots, \psi_n$ are solutions of the homogeneous equation (6) under the following boundary conditions: on all contours, except Γ_k, the function ψ_k is equal to 0; on the contour Γ_k the function ψ_k is equal to 1. By virtue of the unique solvability of the Dirichlet problem for the elliptic equation (6), the functions $\psi_0, \psi_1, \dots, \psi_n$ are completely determined functions and can be found.

Substituting (11) into (10), we arrive at a system of n linear algebraic equations for determining the quantities $Q_1, \dots, Q_n$:

$$\sum_{i=1}^n Q_i \oint_{\Gamma_k} \frac{\partial \psi_i}{\partial \nu} A \, d\sigma_t + \oint_{\Gamma_k} \left(A \frac{\partial \psi_0}{\partial \nu} + F_t \right) d\sigma_t = 0, \qquad k = 1, \dots, n. \tag{12}$$

The determinant of this system is not equal to 0. To prove this, it is sufficient to show that the homogeneous problem for equations (1)–(3) has only the trivial solution. Indeed, let ζ_N, ψ_N be a nontrivial solution of the homogeneous problem*. The function ψ_N satisfies the homogeneous equation (6) and $\psi_N|_{\Gamma_0} = 0$. Since $\psi_N \not\equiv 0$ in the domain D, it is nonzero on at least one inner contour. Choose the contour Γ_l on which the value of ψ_N is greatest. On this contour $\partial\psi_N/\partial\nu < 0$ (^{3}). Since $A > 0$, we have

$$\oint_{\Gamma_l} A \frac{\partial \psi_N}{\partial \nu} \, d\sigma_t \neq 0,$$

which is impossible, since

$$0 = \oint_{\Gamma_l} \frac{\partial \xi_N}{\partial \lambda} \, d\lambda + \frac{\partial \xi_N}{\partial \theta} \, d\theta = \oint_{\Gamma_l} A \frac{\partial \psi_N}{\partial \nu} \, d\sigma_t.$$

After determining $Q_1, \dots, Q_n$ from equations (12), the desired function ψ is formed by formula (11).

Condition (10) can be rewritten in the form (on the basis of relations (5)):

$$\oint_{\Gamma_k} A S_t d\sigma_t = - \oint_{\Gamma_k} F_t d\sigma_t, \qquad k = 1, \dots, n. \tag{13}$$

Relation (13) makes it possible, without solving equations (1)–(3), to determine the direction of the circulation of water near an island. The important role of the nonuniformity of the wind field in determining the direction of circulation around an island was indicated in the work of V. B. Shtokman (^{4}).

In (^{5}) a model of a sea homogeneous in density was proposed for analyzing the features in the distribution of total transports in Antarctica with variable bottom relief. In this case the domain D is doubly connected, and we have only one unknown constant Q (the total transport of the Antarctic circumpolar current). The value of the transport Q is determined by the formula

$$Q = -\frac{\displaystyle\oint_{\Gamma_1} \left(A \frac{\partial \psi_0}{\partial \nu} + F_t \right) d\sigma_t}{\displaystyle\oint_{\Gamma_1} A \frac{\partial \psi_1}{\partial \nu}\, d\sigma_t}. \tag{14}$$

In determining the functions ψ_0, ψ_1 from equation (6), the asymptotic methods set forth in (^{6}) were used. For the value of the coefficient of vertical exchange $\mu_z = 1.4 \cdot 10^2$ cm^2/sec, the value $Q = 1.4 \cdot 10^{14}$ cm^3/sec, which agrees well with observational data (^{7}).

Let us now consider steady currents in a sea inhomogeneous in density. In this case the determination of the current elements at individual levels encounters very great difficulties. However, as V. B. Shtokman showed (^{8}), for total transports in a baroclinic sea the following equations can be written, which we shall write in a spherical coordinate system:

$$A_l \Delta S_\lambda + T_\lambda - 2\omega\rho_0 \cos\theta \cdot S_\theta = g \frac{\partial P}{a_0 \sin\theta\, \partial\lambda}, \tag{15}$$

$$A_l \Delta S_\theta + T_\theta + 2\omega\rho_0 \cos\theta \cdot S_\lambda = g \frac{\partial P}{a_0\, \partial\theta}, \tag{16}$$

$$\frac{\partial S_\lambda}{\partial \lambda} + \frac{\partial}{\partial\theta} (\sin\theta \cdot S_\theta) = 0. \tag{17}$$

Here A_l is the coefficient of horizontal turbulent exchange; T_λ and T_θ are the components of wind stress; $P = \displaystyle\int_0^H dz \int_0^z \rho\, dz$, where ρ is the variable density; ω is the angular velocity of the Earth’ s rotation; ρ_0 is the mean value of the density; Δ is the Laplace operator.

We shall seek solutions S_λ, S_θ, P of equations (15)–(17) in the domain D with boundary conditions

* If S_λ and S_θ are known, then by virtue of equation (3) and condition (4) the function ψ can always be cons

$$S_\nu\big|_{\Gamma_k} = 0, \qquad S_t\big|_{\Gamma_k} = 0, \qquad k = 0, 1, \dots, n. \tag{18}$$

Introduce, as was done in ([8]), the function of the total transports ψ (relations (5)) and form an equation for it by eliminating the function P from equations (15) and (16):

$$A_l \Delta\Delta\psi - \frac{2\omega\rho_0}{a_0}\frac{\partial\psi}{a_0\,\partial\lambda} = \frac{\partial T_\theta}{a_0 \sin\theta\,\partial\lambda} - \frac{1}{\sin\theta}\frac{\partial}{a_0\,\partial\theta}\left(\sin\theta \cdot T_\lambda\right). \tag{19}$$

From relations (18) we obtain the following boundary conditions for ψ:

$$\psi\big|_{\Gamma_0} = 0, \qquad \frac{\partial\psi}{\partial\nu}\bigg|_{\Gamma_0} = 0, \qquad \psi\big|_{\Gamma_k} = Q_k, \qquad \frac{\partial\psi}{\partial\nu}\bigg|_{\Gamma_k} = 0, \qquad k = 1, \dots, n. \tag{20}$$

The conditions for uniqueness of the function P in the domain D impose additional restrictions on ψ. The function ψ must satisfy the following conditions:

$$A_l \oint_{\Gamma_k} \frac{\partial}{\partial\nu}(\Delta\psi)\,d\sigma_t + \oint_{\Gamma_k} T_t\,d\sigma_t = 0, \qquad k = 1, \dots, n. \tag{21}$$

Conditions (21), from a new point of view, reveal the role of lateral friction near the shores of the basin in the case of a baroclinic sea.

By analogy with the case already considered, one can introduce basis functions $\psi_0, \psi_1, \dots, \psi_n$ and represent ψ in the form of a linear combination of these functions. To determine the functions $\psi_0, \psi_1, \dots, \psi_n$, one must solve the first boundary-value problem (on the boundary of the domain the function and its normal derivative are prescribed) for equation (19). After this, conditions (21) give n linear algebraic equations for determining the unknown constants $Q_1, \dots, Q_n$.

In conclusion we note that an analogous method for determining the unknown contour constants $Q_1, \dots, Q_n$ is used in the theory of torsion of rods with multiply connected cross sections ([9]).

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Received
27 XI 1960

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