Mathematics

I. S. Khara

1961

SovietRxiv

Abstract

Full Text

Mathematics

I. S. Khara

On a Numerical Method for Solving Eigenvalue Problems

(Presented by Academician I. G. Petrovsky, 1 VII 1961)

I. Let s in the characteristic equation

$$1 + a_1\lambda + a_2\lambda^2 + \dots + a_s\lambda^s = 0 \tag{1}$$

of a one-dimensional boundary-value problem be such that it ensures the determination of the first s_0 eigenvalues with an error not exceeding several percent. The phenomenon of rapid growth of the quantity $S(s_0) = s - s_0$ as s_0 increases, characteristic of most methods presently known for constructing equation (1), is not characteristic of the proposed method (the latter is clear from the examples given). In determining the eigenvalues of boundary-value problems for the equation

$$q_1(x)y'(x) + q_2(x)y''(x) + \dots + q_n(x)y^{(n)}(x) = \lambda r(x)y(x) \tag{2}$$

with smooth coefficients, we shall obtain the homogeneous system of equations for $\psi_k^{(i)} = y^{(i)}\left(\frac{k-1}{s}\right)$ $(k = 1, 2, \dots, s+1)$ by multiple integration. To replace by finite sums the integrals of the form that arise here,

$$\int_{x_\omega}^{x_{\omega+k}} \int_{x_\omega}^{x} \cdots \int_{x_\omega}^{x} r(x)y(x)\,dx^\nu \tag{3}$$

we introduce for consideration the formulas

$$\int_0^{x_k} \int_0^{x} \cdots \int_0^{x} \varphi(x)\psi(x)\,dx^\nu = b^\nu \sum_{i=1}^{m} A_{ki}^{(\nu,m)}[\varphi]\psi(x_i) + R_k^{(\nu,m)}, \tag{4}$$

$$\int_0^{x_k} \int_0^{x} \cdots \int_0^{x} \psi(x)\,dx^\nu = \frac{b^\nu}{(\nu-1)!} \sum_{i=1}^{m} A_{k,i}^{(1,m)}[1](x_k - x_i)^{\nu-1}\psi(x_i) + \overline{R}_k^{(\nu,m)}; \tag{5}$$

$$A_{ki}^{(\nu,m)}[\varphi] = \sum_{j=1}^{p} B_{k,i,j}^{(\nu,m,p)} \varphi(x_j), \qquad x_j = \frac{j-1}{p-1} b, \qquad x_i = \frac{i-1}{m-1} b, \qquad (6)$$

$$B_{k,i,j}^{(\nu,m,p)} = b^{-\nu} \int_0^{x_k} \int_0^x \cdots \int_0^x l_j^{(p)}(x) l_i^{(m)}(x)\, dx^\nu, \qquad k = 2, 3, \dots, m,$$

where $l_j^{(p)}(x)$ and $l_i^{(m)}(x)$ are the coefficients of the Lagrange polynomial for the nodes x_j and x_i. Before making definite recommendations for computing the integrals (3), we shall carry out a comparative analysis of formulas (4) and (5). Let us first compare formula (5) with the particular case of formula (4) for $\varphi(x) = 1$,

which we shall denote by (4a), and then formula (4a) with formula (4). Comparing formulas (5) and (4a) (it is assumed that $\nu > 1$; for $\nu = 1$ formulas (4a) and (5) coincide) makes sense only under definite restrictions imposed on the function $\psi(x)$. It is easy to indicate particular cases in which either of the indicated formulas will have an advantage over the other. Thus, for example: 1) for $\psi(x) = (b-x)^{-1/2}$ formula (4a) is inapplicable, while formula (5) is applicable; 2) for $\psi(x) = (b-x)^{1/2}$ formula (5) may give a better result than formula (4a); 3) for $m = 3,\ k = 2,\ \psi(0) = \psi(b) = 0$, formula (5) will give an identically zero answer, while the result given by formula (4a) may be satisfactory. For the technical practice of determining eigenvalues, cases 1) and 2) are not typical, whereas case 3) is typical. If one has in mind the application of formulas (4a) and (5) to functions $\psi(x)$ having a sufficient number of smooth derivatives, then the advantage of the first of them over the second is obvious (the degree of accuracy of formula (4a) is $\nu - 1$ units higher than the degree of accuracy of formula (5)). The advantage of formula (4a) over formula (5) for the indicated class of functions $\psi(x)$ may also be established by comparing the remainder terms. Thus, for example, we have

$$R_2^{(3,3)}[\psi] = -\frac{b^6}{9216}\, \psi'''(\xi_2),$$

$$\overline{R}_2^{(3,3)} = \frac{1}{2!}\, R_2^{(1,3)} \left[\left(\frac{b}{2} - x \right)^2 \psi(x) \right] = \frac{b^4}{768} \left[\left(\frac{b}{2} - x \right)^2 \psi(x) \right]_{x=\zeta_2}''',$$

where ξ_1 and ξ_2 are certain numbers from the interval $[0, b]$. The advantage of using formula (4) in comparison with formula (4a) can be illustrated by the following simple example. Suppose that in integral (3) $r(x)$ and $y(x)$ behave approximately as x^2 and $\sin x$.

Table 1

$$A_{k,i}^{(\nu,m)}[1]; \quad m = 3$$

	$\nu=2$	$\nu=2$	$\nu=3$	$\nu=3$	$\nu=4$	$\nu=4$
	$k=2$	$k=3$	$k=2$	$k=3$	$k=2$	$k=3$
$i=1$	7	1	27	9	11	8
$i=2$	6	2	16	12	5	8
$i=3$	-1	0	-3	-1	-1	-1
Denominators	96	6	1920	120	5760	360

The error in computing integral (3) depends: 1) when computing by formula (4) with $\varphi(x) = r(x)$, $\psi(x) = y(x)$, on the values of the m-th derivative of $\sin x$; 2) when computing by formula (4a), on the values of the m-th derivative of $x^2 \sin x$. The computation of repeated integrals by formula (5) occurs in the book ([1]) (§§ 10, 51, 55, etc.), and detailed tables $A_{k,i}^{(1,m)}[1]$ are available in the book ([2]) (Tables 65-72). Below we give a list of recommended coefficients for solving eigenvalue problems

$$A_{k,i}^{(\nu,m)}[x^\gamma] \quad (m = 3, 4;\ \gamma = 0, 1, 2;\ \nu = 1, 2, 3, 4);$$

$$B_{k,i,j}^{(\nu,m,p)} \quad (\nu = 1, 2, 3, 4;\ m = 3, 4;\ p = 3, 4, 5).$$

All the indicated coefficients, whose total volume does not exceed 1/4 of a printed sheet, are computed once and for all. For solving many types of problems, the coefficients $A_{k,i}^{(\nu,m)}[1]$ prove sufficient; part of them is given in Table 1.

We give brief recommendations for computing the integrals (3). If $r(x) = x^\gamma$ $(\gamma = 0, 1, 2)$, then in (4) we set $\varphi(x) = x^\gamma$, $\psi(x) = y(x)$ and compute the integrals with the aid of the coefficients $A_{k,i}^{(\nu,m)}[x^\gamma]$. If $r(x)$

different from x^γ, but varies slowly, then we compute the integrals by formula (4a) with $\psi(x) = r(x)y(x)$. If $r(x)$ is different from x^γ and varies sufficiently rapidly, then we compute the integrals by formula (4) with $r(x) = \varphi(x)$, $\psi(x) = y(x)$, while the coefficients $A_{k,i}^{(\nu,m)}[\varphi]$ are first computed by formula (6) with $\varphi(x) = r(x)$. We note that the effort expended on computing the coefficients $A_{k,i}^{(\nu,m)}[\varphi]$ by formula (6) is, as a rule, justified.

II. Let the boundary conditions $y'(0) = y(1) = 0$ be imposed for equation (2) with $n = 2$, and suppose it is required to construct equation (1) for sufficiently large s. Integrating equation (2) twice from x_k to x and replacing the integrals for $x = x_{k+1}$ and $x = x_{k+2}$ by finite sums by means of formula (4), we obtain two equations; by eliminating the quantity y'_k from them we arrive at the relation

$$y_{k+2}D^{(2)}_{k+2} = y_{k+1}D^{(1)}_{k+1} + y_{kD}k^{(0)}. \tag{7}$$

Both equations from which relation (7) is formed, for $k = 1$, do not contain derivatives. Adjoining one of them to system (7), we obtain s homogeneous equations with s unknowns. We shall divide all the computational work in constructing equation (1) into two stages: 1) replacement of the integrals by finite sums, formation of relation (7), reduction of the coefficients $D^{(2)}_{k+2}$ to the form $1 + \lambda d^{(2)}_{k+2}$, and expression of y_2 and y_3 in terms of λ (we set y_1 equal to unity); 2) successive expression of the remaining y_k in terms of λ by means of relation (7). Equating y_{s+1} to zero, we arrive at equation (1). Taking into account the computational work only at the second stage (for large s it is the dominant one), we arrive at the formulas for the number of arithmetic operations: 1) the number of multiplications $H_1(s) = 3s^2 - s - 12$; 2) the number of additions $H_2(s) = 2.5s^2 - 0.5s - 7$. With the determinant method of constructing equation (1), the number of arithmetic operations will be of order 4^s. Since taking account of other types of linear boundary conditions both for equation (2) and for equations of higher orders presents no difficulty, without dwelling on this question we shall pass to the consideration of numerical examples; moreover, we shall not write out equation (1), but shall report only the results of solving this equation, denoting its roots by $\lambda^{(s)}_k$, where k is the number of the root and s is the order of equation (1).

Example 1. For $((1 + x)y')' = -\lambda(1 + x)y,\ y'(0) = y(1) = 0$, we have:

1) by the finite-difference method ((3), p. 106) $\lambda^{(5)}_1 = 3.15$ (−2%), $\lambda^{(5)}_2 = 21.11$ (−8%);
2) by the proposed method $\lambda^{(2)}_1 = 3.23$ (0.3%), $\lambda^{(2)}_2 = 32.2$ (40%), $\lambda^{(3)}_2 = 22.85$ (−0.6%), $\lambda^{(3)}_3 = 87$ (40%), $\lambda^{(5)}_3 = 62.5$ (0.3%), $\lambda^{(5)}_4 = 124.2$ (1.2%), $\lambda^{(5)}_5 = 207$ (3%).

Example 2. For $y'' = -\lambda xy,\ y(0) = y(1) = 0$: 1) in work [(4)] the result $\lambda^{(7)}_1 = 18.955$ (0.05%) was obtained; by the method of the same author [(4)] we find $\lambda^{(3)}_1 = 17.4$ (−8%); 2) by the method of the author [(1)] (p. 52) $\lambda^{(4)}_1 = 18.25$ (−3.7%); 3) by the proposed method $\lambda^{(1)}_1 = 19.2$ (1.3%); $\lambda^{(2)}_1 = 18.89$ (−0.3%).

Example 3. For $M'''(t) = -\lambda\left[\alpha^2(1 + \alpha^2t^2)^{3/2}M'(t) + 4\alpha^4t(1 + \alpha^2t^2)^{1/2}M(t)\right]$, $\alpha = 0.4,\ M(0) = M''(0) = M(1) = 0$, we have: 1) by the author' s method [(1)] (p. 223) $\lambda^{(3)}_1 = 56.16$ (−0.4%); 2) by the proposed method $\lambda^{(2)}_1 = 56.17$ (−0.3%).

III. As is known, the presence of concentrated masses, even at a small number of points, sharply increases the labor involved in computing the flexural vibrations of rods. In connection with what has been said, the question naturally arises: is it possible to circumvent the difficulties caused by the presence of concentrated masses? If at the interior points η_i ($i = 1, 2, \dots, N$) of the interval $[0 \le x \le 1]$ there are concentrated masses M_i, then the differential equation of flexural

the vibrations of the rod can be written in the form

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2y(x)}{dx^2}\right] = p^2\left[m(x) + \sum_{i=1}^{N} M_i(x)\right] y(x), \tag{8}$$

where

$$\int_{\eta_i-h}^{\eta_i+h} M_i(x)\,dx = M_i,$$

and h is a small quantity.

If the boundary-value problem for equation (8) is solved by repeated integration, then to the former unknowns $y_k^{(i)} = y^{(i)}\left(\frac{k-1}{s}\right)$ (not all derivatives and not at all points are included among the unknowns) new unknowns $y(\eta_i)$ are added. Although it seems natural to set up the missing equations, a substantially more rational approach is another one—the reduction of the number of unknowns; namely, each ordinate $y(\eta_i)$ can be expressed, using tables of interpolation coefficients, in terms of the three or four ordinates $y_k^{(0)}$ nearest to it. The effectiveness of the indicated interpolation has been verified on a number of examples.

Kharkov Polytechnic Institute
named after V. I. Lenin

Received
28 VI 1961

REFERENCES CITED

1. Sh. E. Mikeladze, *New Methods for Integrating Differential Equations*, 1951.

2. Sh. E. Mikeladze, *Numerical Methods of Mathematical Analysis*, 1953.

3. L. Collatz, *Numerical Methods for Solving Differential Equations*, 1953.

4. I. M. Rapoport, *Ukr. Math. J.*, 7, No. 2 (1954).

Note: Figure translations are in progress. See original paper for figures.