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For the equation

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Abstract

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CERTAIN BOUNDARY-VALUE PROBLEMS FOR ELLIPTIC EQUATIONS WITH STRONG DEGENERATION AT THE BOUNDARY

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For the equation

$$\frac{\partial}{\partial x}\left[y^k\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[y^k\frac{\partial u}{\partial y}\right]=0, \tag{1}$$

considered in a bounded simply connected domain G with piecewise smooth boundary Γ, containing a segment Γ_0 of the axis Ox, the formulation of boundary-value problems depends essentially on the value of k. From the results of M. V. Keldysh ([1]) it follows that, in the class of continuous boundary data and solutions bounded in $\overline{G}$, for equation (1) either the Dirichlet problem is uniquely solvable when $k<1$, or problem E is uniquely solvable when $k>1$ (in which the segment Γ_0 is freed from boundary conditions). Analogous results were obtained by other authors (see the monograph ([2]) and the bibliography therein), proceeding from a variational treatment of the corresponding problems. In the present paper we investigate other formulations of boundary-value problems for equations of type (1), originating with A. V. Bitsadze ([3]).

In a domain G of the indicated kind, consider the equation

$$L(u)=\frac{\partial}{\partial x}\left[\sigma^k(x,y)\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[\sigma^k(x,y)\frac{\partial u}{\partial y}\right]=0, \tag{2}$$

where $\sigma(x,y)$ is a sufficiently smooth positive function satisfying the estimate

$$c_1y<\sigma(x,y)<c_2y, \qquad c_1,c_2>0.$$

Thus equation (2) degenerates on the segment Γ_0. We shall call the degeneration strong (weak) for $k\ge 1$ ($k<1$); the case $k=1$ will also be called critical. Let first $k>1$.

Problem A. Find a solution $u(x,y)$, twice continuously differentiable in the domain G, of equation (2), which assumes on the boundary Γ, in the mean, the values

$$\lim_{(x,y)\to M\in\Gamma}\left[\sigma^{k-1}(x,y)u(x,y)\right]=\varphi(M). \tag{3}$$

Theorem 1. Suppose that Γ does not touch the axis Ox and contains no reentrant corners. If $1<k<2$, the function $\sigma(x,y)$ is four times boundedly differentiable and $\Delta\sigma\ge 0$, then problem A has a unique solution provided the requirements

$$\text{a)}\ \ \varphi(M)\in L_2(\Gamma);\qquad \text{b)}\ \ \int_\Gamma ds_M\int_\Gamma \frac{|\varphi(M)-\varphi(Q)|^2}{|MQ|^2}\,\omega^{2-k}(M,Q)\,ds_Q<\infty, \tag{4}$$

are fulfilled, where $\omega(M,Q)$ is equal to the distance $|MQ|$ between the points M and Q, if at least one of these points lies on Γ_0, and is equal to the lesser of the distances from these points to the axis Ox otherwise.

For the proof, consider the auxiliary functional

$$I(v)=\iint_G\left\{\sigma^{2-k}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+(k-1)\Delta\sigma\cdot\sigma^{1-k}v^2\right\}dx\,dy \tag{5}$$

in the class $W(\varphi)$ of functions from $W^{(1)}_{2,\,2-k}$ (see ([2])), taking on the boundary the prescribed values $\varphi(M)$. Conditions (4) constitute a characterization (a necessary and sufficient criterion) of the boundary values of functions from $W^{(1)}_{2,\,2-k}$, and therefore the class $W(\varphi)$ is nonempty. On the basis of arguments customary for the variational method ([4]), we are convinced of the existence of a unique element $v_0\in W(\varphi)$ minimizing the functional $I(\varphi)$. The function v_0, by virtue of the conditions of the theorem, turns out to be twice continuously differentiable ([2]), satisfies the Euler-Lagrange equation of the functional (5), and, in the mean quadratic sense, the boundary condition

$$\lim_{(x,y)\to M\in\Gamma} v_0(x,y)=\varphi(M).$$

From the uniqueness theorem of paper ([5]) (p. 263) it follows that there can be no other solution with the indicated properties. By a direct verification one may now ascertain that the function $u_0=v_0/\sigma^{k-1}$ satisfies equation (2) and is a solution of Problem A. It may also be asserted that for the function u_0 the integral $I(\sigma^{k-1}u_0)$ is finite.* The theorem is proved.

Let us note that the restrictions $k<2$ and those on σ have been imposed here by the method of proof. Problem A has meaning also under broader assumptions, in particular in the case of strong degeneration along the entire boundary. As a useful example, consider Problem A for equation (1) in the half-plane $y > 0$. After the substitution $y^{k-1}u = v$, equation (1) takes the form

$$\frac{\partial}{\partial x}\left(y^{2-k}\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(y^{2-k}\frac{\partial v}{\partial y}\right)=0,$$

and the solution of the posed problem for all $k > 1$ is given by the formula

$$u_0(x,y)=\frac{1}{\sqrt{\pi}}\frac{\Gamma(k/2)}{\Gamma((k-1)/2)}\int_{-\infty}^{\infty}\frac{\varphi(\xi)\,d\xi}{[(x-\xi)^2+y^2]^{k/2}}. \tag{6}$$

The uniqueness of this solution for $1 < k < 3$ can be obtained from the requirement that $D_{2-k}(y^{k-1}u_0)$ be finite; here the boundary function $\varphi(x)$ must be taken from $W_2^{(k-1)/2}(-\infty,\infty)$. These assertions follow immediately from the results of paper ([6]).

The case of critical degeneration $k = 1$ is of interest. For simplicity we shall restrict ourselves to the case $\sigma = y$ and pose the problem as follows:

Problem B. Find in the domain G a twice continuously differentiable solution of the equation

$$L_1(u)=y\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial y}=0,$$

which takes on the boundary of the domain, in the mean sense, the values

$$\left[\frac{1}{\ln\dfrac{M}{y}}\,u(x,y)\right]_{(x,y)\to P\in\Gamma}=\varphi(P)$$

and has finite integral

$$\iint\limits_G y\ln^2\frac{M}{y}\left\{\left[\frac{\partial}{\partial x}\left(\frac{1}{\ln\dfrac{M}{y}}\,u\right)\right]^2+\left[\frac{\partial}{\partial y}\left(\frac{1}{\ln\dfrac{M}{y}}\,u\right)\right]^2\right\}dx\,dy,$$

where M is a constant exceeding the diameter of the domain.

* If one requires in advance of the solution the finiteness of the integral $I(\sigma^{k-1}u)$, then conditions (4) for the

Let the domain G satisfy the same requirements as in Theorem 1, and let $\omega(P,Q)$ have the same meaning.

Theorem 2. For Problem B to have a solution, it is necessary and sufficient that the boundary function $\varphi(P)$ satisfy the conditions

$$\text{a)}\ \ \varphi(P) \in L_2(\Gamma); \qquad \text{b)}\ \ \int_\Gamma ds_P \int_\Gamma \frac{|\varphi(P)-\varphi(Q)|^2}{|PQ|^2}\,\omega(P,Q)\,ds_Q < \infty. \quad (7)$$

Proof. Consider the functional

$$D(v,G) = \iint_G y \ln^2 \frac{M}{y} \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \right] dx\, dy \quad (8)$$

on the class $\hat{D}(G)$ of functions for which it is defined and finite. First of all we establish that a function $v(x,y) \in \hat{D}(G)$ has, in the sense of convergence in the mean, a boundary value $F(P)$ with properties (7); and, conversely, a function $F(P)$ given on Γ and possessing properties (7) can be extended to G in the weighted class $\hat{D}(G)$. This is done according to the schemes indicated in [7], except that instead of Hardy' s inequality one uses the inequality

$$\int_0^a \frac{dx}{x} \left| \int_0^x f(t)\,dt \right|^p \le p^p \int_0^a x^{p-1} \ln^p \frac{M}{x}\, |f(x)|^p\, dx, \qquad p \ge 1,\ M \ge a.$$

Then the problem of finding a function v_0 from $\hat{D}(G)$, possessing the boundary value $\varphi(P)$, on which the functional (8) attains its least value, is considered. Similarly to how this was done in Theorem 1, the existence of a unique function v_0 with the indicated properties is proved, and, just as there, it is established that the function $u_0 = \ln \frac{M}{y} \cdot v_0$ is the unique solution of Problem B.

We note that in the case of the half-disk $\{x^2+y^2<R^2,\ y>0\}$ the solution of Problem B can be written in explicit form, using the Green function of the operator $L_1(u)$:

$$\Gamma(x,y;\xi,\eta) = g(x,y;\xi,\eta) - \frac{R}{\rho}\, g(x,y;\xi^*,\eta^*),$$

$$g(x,y;\xi,\eta) = \frac{1}{2\pi} \frac{1}{\ln\frac{M}{y}\,\ln\frac{M}{\eta}} \int_0^\pi \frac{d\alpha}{[(x-\xi)^2+y^2+\eta^2-2y\eta\cos\alpha]^{1/2}},$$

$$\frac{\xi^*}{\xi} = \frac{\eta^*}{\eta} = \frac{R^2}{\rho^2}, \qquad \rho^2 = \xi^2+\eta^2.$$

Then, by a limiting passage, one can obtain the solution of Problem B in the half-plane in the form

$$u(x,y) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\varphi(t)\,dt}{\left[(x-t)^2+y^2\right]^{1/2}}. \tag{9}$$

It is easy to verify that for continuous functions $\varphi(x)$ that decrease sufficiently rapidly at infinity, the function (9) satisfies the condition

$$\lim_{y\to 0} \frac{u(x,y)}{\ln(1/y)} = \varphi(x).$$

The examples given lead to the following remark. It is known that the function (9) is the spatial potential of charges distributed on the axis Ox with density $\varphi(x)$ (in this case one should

consider y as the distance to the axis Ox. In exactly the same way, interpreting, for $k > 1$, equation (1) as the heat-propagation equation with thermal-conductivity coefficient y^α ($k = 1+\alpha$), formula (6) may be interpreted as the stationary distribution of temperature in space, created by sources distributed along the axis Ox with density $\varphi(x)$. Consequently, the formulations of boundary-value problems considered here make it possible to single out solutions of the corresponding equations that are singular on the boundary, and have a simple physical meaning.

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Note: Figure translations are in progress. See original paper for figures.