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Abstract

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ON PERFECT BICOMPACT EXTENSIONS

(Presented by Academician P. S. Aleksandrov, 14 X 1960)

Let X be a completely regular space. A bicompact extension Y of the space X is called **perfect with respect to an open set** $U \subset X$ if the closure in Y of the boundary $\operatorname{Fr}_X U$ of the set U in X is the boundary in Y of the set $O\langle U \rangle^*$. The extension Y is called simply **perfect** if it is perfect with respect to every open set of the space X.

Lemma 1. *Let δ be the proximity relation on the space X corresponding to the bicompact extension Y. The bicompact extension Y of the space X is perfect with respect to the open set $U \subset X$ if and only if, for every set $A \subset U$, from $\bar{A}\delta \operatorname{Fr}_X U$ it follows that $\bar{A}\delta(X \setminus U)$.*

Corollary. *The Čech extension βX of a completely regular space X is a perfect bicompact extension**.*

The following theorem shows that one can give several mutually equivalent definitions of perfect bicompact extensions. As usual, we shall say that a closed set F **splits** the space X into the sets U_1 and U_2 if $X \setminus F = U_1 \cup U_2$, where U_1 and U_2 are nonempty open sets in X, and $U_1 \cap U_2 = \Lambda$. We shall say that a set N (not necessarily closed) **splits** the space Y at a point $y \in N$ if this point has a neighborhood U such that $U \cap (Y \setminus N) = V' \cup V''$, where V' and V'' are open in $Y \setminus N$, with $V' \cap V'' = \Lambda$ and $y \in Y[V'] \cap Y[V'']$.

Theorem 1. *Let Y be a bicompact extension of a completely regular space X. The following properties of the extension Y are equivalent:*

1) *the extension Y is perfect;*
2) *the remainder $Y \setminus X$ does not split the bicompactum Y at any of its points**;*
3) *for any two disjoint open sets U_1 and U_2 the relation*** holds*

$$O\langle U_1 \cup U_2 \rangle = O\langle U_1 \rangle \cup O\langle U_2 \rangle;$$

4) *if a closed set F splits the space X into the sets U and V, then the set $Y[F]$ splits the bicompactum Y into the sets $O\langle U \rangle$ and $O\langle V \rangle$****.*

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* $O_Y\langle U\rangle$ denotes the largest open set of the space Y that cuts out the set U on the subspace X, i.e. $O\langle U\rangle = Y \setminus Y[X \setminus U]$. In what follows, the terminology and results from article (6) will be widely used.

** This result belongs to Yu. M. Smirnov (5).

*** In application to the Čech extension, this property may be regarded as a generalization of the following result of Henriksen and Isbell (7): an open set U in the Čech extension βX of a space X is connected if and only if the set $U \cap X$ is connected. This fact, thus, is valid for every perfect extension.

**** We note that this relation holds for any two open sets if and only if the space is normal and the extension is the Čech extension.

***** Here one cannot discard the requirement that $Y[F]$ split Y precisely into the sets $O\langle U\rangle$ and $O\langle V\rangle$, since otherwise there would exist imperfect bicompact extensions satisfying this condition.

Theorem 2. In order that a bicompact extension Y of a space X be perfect, it is necessary and sufficient that the natural mapping of the Čech extension βX onto the extension Y be monotone.

The assertion of Theorem 2 is contained in the following two lemmas.

Lemma 2. Let Y be a perfect bicompact extension of a space X, and let Z be an arbitrary bicompact extension of the same space following the extension Y; then the natural mapping $\varphi : Z \to Y$ is monotone.

Lemma 3. Let Z be a perfect bicompact extension of a space X, and let Y be a bicompact extension of the same space preceding the extension Z, and suppose that the natural mapping $\varphi : Z \to Y$ is monotone; then the extension Y is perfect.

A space N is called **punctiform** if every connected bicompact subset of it reduces to a single point. From Theorem 2 and Lemmas 2 and 3 the following theorem follows.

Theorem 3. A space X has a minimal perfect extension if and only if it has at least one bicompact extension with punctiform remainder. In this case the minimal perfect extension μX is unique, has a punctiform remainder, and is maximal among all bicompact extensions with punctiform remainder.

An obvious consequence of Theorems 1 and 3 is the following strengthening of Duda' s theorem ([3]) on the extension of homeomorphisms to bicompact extensions:

Let Y_1 and Y_2 be bicompact extensions of spaces X_1 and X_2, respectively, such that the remainders $Y_1 \setminus X_1$ and $Y_2 \setminus X_2$ are punctiform and do not cut any of the bicompacts Y_1 and Y_2 at any point. Then every homeomorphism (if such exists) between the spaces X_1 and X_2 extends to a homeomorphism between the bicompacts Y_1 and Y_2.*

Theorem 3 shows that a space has a unique perfect extension if and only if the remainder in its Čech extension is punctiform. For metrizable spaces this leads to the following result:

Theorem 4. The Čech extension of a metrizable space X is the unique perfect bicompact extension if and only if $X = \Phi \cup U$, where Φ is compact and $\dim U = 0$.

In proving this theorem the following lemma is used.

Lemma 4. Let X be a normal space, and let X_n be the set of points $x \in X$ such that $\operatorname{loc}\dim_x X \geq n$.** If the set X_n is not compact, then the remainder in the Čech extension of the space X contains a bicompact of dimension $\geq n$ (in the sense of dim).

Now let X be a peripherally bicompact Hausdorff space. A **π-bicompact base** on such a space is a base of open sets such that: 1) the boundaries of the sets of this base are bicompact; 2) with every set U it contains $X \setminus X[U]$, and with any two sets U_1 and U_2 it contains the sets $U_1 \cap U_2$ and $U_1 \cup U_2$. To every π-bicompact base there naturally corresponds a proximity relation on the space X, and the bicompact extension corresponding to this proximity relation has a zero-dimensional (in the sense of ind) remainder ([4]). Such bicompact extensions (i.e. those corresponding to π-bicompact bases) will for brevity be called **π-extensions**. Different π-bicompact bases may correspond to one and the same π-extension. To avoid this, one must consider π-bicompact bases satisfying the following additional condi-

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* The difference between this formulation and Duda’ s theorem consists, first, in the fact that there, if our notation is used, Y_1 and Y_2 are compacta, and, second, in the fact that the condition of “non-cutting” of the bicompacts by the remainders is understood here in a less restrictive sense.

** For the definition of local dimension see ([2]).

definition: let U be an open set of the space X with bicompact boundary such that for every closed set $A \subset U$ there exists a set V from the base containing A and contained in U; then the set U itself must belong to the base. The correspondence between such π-bicompact bases and π-extensions is one-to-one.

Among the π-extensions of the space X, there is, obviously, a unique maximal one—the extension corresponding to the π-bicompact base consisting of all open sets with bicompact boundary.

Theorem 5. *The maximal π-extension is perfect.*

It follows directly from Theorems 3 and 5 that the maximal π-extension of a peripherally bicompact space X coincides with the minimal perfect extension μX and, consequently, is maximal among extensions with pointiform (in particular, with zero-dimensional) remainder.

The following two theorems give, in a certain sense, a description of the bicompact extensions of a peripherally bicompact space that follow the maximal π-extension and that precede it.

Theorem 6. *A bicompact extension Y of a peripherally bicompact space X follows the maximal π-extension μX if and only if it is perfect relative to all open sets of the space X that have bicompact boundary.*

Theorem 7. *A bicompact extension Y of a peripherally bicompact space X precedes the maximal π-extension μX if and only if it corresponds to some uniform structure of finite open covers consisting of sets with bicompact boundary.*

In conclusion we apply the preceding results to obtain an answer to the following question from the survey of P. S. Aleksandrov (1): does every peripherally bicompact space possess a bicompact extension with zero-dimensional remainder having the same dimension as the space itself? The answer turns out to be negative. Let X be the set of points of the unit square in the plane for which at least one coordinate is rational, and let Y be the closure of the space in the plane (i.e., the unit square). From Theorems 1 and 3 it follows that all bicompact extensions of the space X with zero-dimensional remainder precede the extension Y, whence, in turn, it follows that all of them have dimension $\geqslant 2$, whereas the dimension of the space X is equal to 1.

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References

1. P. S. **Aleksandrov**, UMN, 15, 2, 25 (1960).
2. C. H. **Dowker**, Quart. J. Math., 6, 101 (1955).
3. R. **Duda**, Indagationes Math., 22, 2, 132 (1960).
4. E. **Sklyarenko**, DAN, 120, No. 6, 1200 (1958).
5. Yu. M. **Smirnov**, "Matem. sborn." , 29, 1, 157 (1951).
6. Yu. M. **Smirnov**, Matem. sborn., 31, 3, 543 (1952).
7. M. **Henriksen**, J. R. **Isbell**, Illinois J. Math., 1, 574 (1957).

Note: Figure translations are in progress. See original paper for figures.