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Abstract

Full Text

MATHEMATICS

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LOCALLY TOPOLOGICALLY COMPLETE SPACES*

(Presented by Academician P. S. Aleksandrov, 14 X 1960)

Definition. We shall call a space** P **locally complete** if every point $x \in P$ is contained in some open complete*** subspace of the space P .

Obviously, every complete space is locally complete. The converse assertion is false:

Example 1. Let T be the set of all countable ordinal numbers, and let K be a segment of the number line. Order the set $R = T \times K$ lexicographically. In the topology determined by this order, R is a compact, locally bicomact, and locally metrizable space. The set R' of all $(a, 0)$, where a is an isolated number, is not an F_σ -set in R , because only finite $M \subset R'$ are closed in R . Thus, P is not a G_δ -set in R , and since P is dense in R , P is not complete. Obviously, P is locally complete and locally metrizable.

Theorem 1. Every paracompact locally complete space P is complete.

Proof. By definition there exists an open cover \mathfrak{A} consisting of complete subspaces. Since P is paracompact, one can inscribe in \mathfrak{A} a locally finite open cover \mathfrak{B} . For every $B \in \mathfrak{B}$ there exists a complete⁽⁶⁾ sequence $\{\mathfrak{A}_n(B)\}$ of open covers of the space B such that $\mathfrak{A}_{n+1}(B) \subset \mathfrak{A}_n(B)$. Denote by \mathfrak{A}_n the union of all $\mathfrak{A}_n(B)$, $B \in \mathfrak{B}$. It is not difficult to prove that the sequence $\{\mathfrak{A}_n\}$ is complete. Thus, P is complete.

Theorem 2. Every locally complete space contains a dense, open and complete space.

Proof. If \mathfrak{A} is some maximal disjoint subsystem of the system of all complete open subspaces, then the union of the system \mathfrak{A} is complete, dense and open.

Obviously, every locally bicomact space is a k -space****.

However, a locally complete space need not be a k -space:

Example 2. Let N be an infinite countable discrete space. Take a sequence $\{N_n\}$ of subsets of the set N such that the sets $N_n - N_{n+1}$ are infinite. Denote by K_n the closure

* Example 1 and Theorem 1 of the present paper are also contained in A. Arkhangel'skii's paper "On topological spaces complete in the sense of Čech" (⁵), submitted for publication on 18 V 1960. P. S. Aleksandrov.

** "Space" always means a completely regular topological space.

*** We call a space P **complete** if it is topologically complete in the sense of E. Čech, i.e. if for every space $R \supset P$ such that $\overline{P} = R$, P is a G_δ -set in R .

**** A space P is called a k -space if every set $M \subset P$ whose intersection with every bicomact $K \supset P$ is closed is itself closed.

sets N_n in the Čech extension $\beta(N)$ of the space N . It turns out that the complete subspace $P = N \cup \bigcap_{n=1}^{\infty} K_n$ of the space $\beta(N)$ is not a k -space.

Theorem 3. If an open subset U of a locally complete space P contains some point of condensation of a countable $N \subset P$, then there exists a bicomact set $K \subset U$ intersecting N in an infinite set.

Proof is easily carried out using the characterization of complete spaces by means of complete sequences of open coverings.

From this it follows easily:

Theorem 4. The topological product of a locally complete compact space and any compact space is compact.

Theorem 5. If an open subset U of a locally complete space P contains some point of condensation of a countable disjoint system \mathfrak{A} of open subsets of the space F , then there exists a bicomact $K \subset U$ intersecting infinitely many elements of the system \mathfrak{A} .

From this it easily follows:

Theorem 6. The topological product of a pseudocompact locally complete space and any pseudocompact space is pseudocompact.

The **character** of a point x of a space P is called (¹) the least cardinality of a complete system of neighborhoods of the point x . The **pseudocharacter** of a point x of a space P is called (¹) the least cardinality of a system \mathfrak{A} of open sets for which $\bigcap \mathfrak{A} = \{x\}$.

Theorem 7. In a locally complete space, the characters and pseudocharacters of points coincide.

A mapping of a space P into a space Q will be called **open** if the images of open subsets of the space P are open in Q .

Theorem 8. Let f be a continuous and open mapping of a space P onto a space Q . If P is locally bicomact, complete, or locally complete, then Q , respectively, is locally bicomact, complete, or locally complete*.

The first assertion is obvious; the third is a consequence of the second, which we shall now prove. Let F be the continuous extension of the mapping f to the

Čech extension $\beta(P)$. Since P is complete, $P = \bigcap_{n=1}^{\infty} U_n$, where U_n are open subsets of the space P . It is not difficult to prove that the open kernel V_n of the set $F[U_n]$ contains Q and $Q = \bigcap_{n=1}^{\infty} V_n$.

Theorem 9. Let f be a perfect (i.e., continuous, closed, and such that the complete inverse images of all points are bicomact) mapping of a space P onto a space Q . In order that the space P be locally bicomact, complete, or locally complete, it is necessary and sufficient that the space Q be, respectively, locally bicomact, complete, or locally complete.

Proof. Let F be the mapping from the proof of Theorem 8. It is not difficult to prove that $F^{-1}[Q] = P$. Consequently, if Q is locally bicomact or complete, then P also is, respectively, locally bicomact or complete. If $M \subset Q$, then $f|f^{-1}[M]$ is a perfect mapping of the space $f^{-1}[M]$ onto M . Thus, by what has already been proved, if Q is locally complete, then P is also locally complete.

* This result, even for multivalued perfect mappings, was proved by V. Ponomarev ⁽²⁾ (for locally bicomact spaces and for complete spaces).

Now suppose that P is complete. Let $\{\mathfrak{A}_n\}$ be a complete sequence of additive open covers of the space P . Denote by \mathfrak{B}_n the system of all $Q - f[P - A]$, where $A \in \mathfrak{A}_n$. It is not difficult to prove that $\{\mathfrak{B}_n\}$ is a complete sequence of the space Q . Thus, Q is a complete space.

Finally, suppose that the space P is locally complete. Every point $x \in P$ is contained in some open and complete $U(x) \subset P$. Let $y \in Q$. Denote the set $f^{-1}[y]$ by K . Since K is bicomact, there exists a finite set $M \subset P$ such that

$$U = \bigcup \{U(x); x \in M\} \supset K.$$

Put $V = Q - f[P - U]$. Clearly, the set V is open and

$$H = f^{-1}[V] \subset U.$$

Since U is complete, H is also complete, and, finally, by what was proved above, V is complete. The assertion on local bicomactness is proved quite analogously.

From Theorems 7 and 9 it follows:

Theorem 10. Let f be a closed and continuous mapping of a complete metrizable space P onto a space Q . In order that Q be metrizable, it is necessary and sufficient that Q be a complete space.

Proof. Since f is a closed mapping, the pseudocharacters of the points of the space Q are countable. Thus, if Q is complete, then, by Theorem 7, the characters of the points of the space Q are countable. From the countability of the characters of the points of the space Q and from the properties of the mapping f , it follows by a well-known theorem of Stone ⁽³⁾ that Q is a metrizable space.

If Q is a metrizable space, then, by a well-known theorem of I. A. Vainshtein ⁽⁴⁾, the boundaries of the inverse images of points are compact. For every $y \in Q$, let $K(y)$ be the boundary of the set $f^{-1}[y]$, if it is nonempty; if it is empty, let $K(y)$ be some one-point subset of the set $f^{-1}[y]$.

Clearly, the subspace

$$K = \bigcup \{K(y); y \in Q\}$$

is complete, and the mapping f , considered on K , is perfect. Thus, by Theorem 9, the space Q is complete.

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