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Abstract

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ON THE EQUIVALENCE OF CERTAIN UNBOUNDED OPERATORS IN A BANACH SPACE

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Let L be a linear unbounded operator acting in a Banach space X, with domain of definition $D(L)$ and closed range $R(L)$, possessing the following properties:

1)
$$R(L) \supset \prod_{k=1}^{\infty} D(L^k).$$

2) The subspace X_0 of solutions of the equation $Lx = 0$ has finite dimension $n > 0$.

3) The operator L_0^{-1}, defined as follows, is bounded. Let $f_0, f_1, \dots, f_{n-1}$ be some system of linearly independent elements of the subspace X_0, and let X_1 be the closure, in the norm $\|x\|_1 = \|x\| + \|Lx\|$, of the domain $D(L)$. Further, let $\varphi_0, \varphi_1, \dots, \varphi_{n-1}$ be a system of bounded linear functionals on the space X_1, biorthogonal to the basis $f_0, f_1, \dots, f_{n-1}$. Then for every $y \in R(L)$ there exists a unique element $x \in D(L)$ such that $Lx = y$, $\varphi_s(x) = 0$ $(s = 0, 1, \dots, n-1)$. We put $x = L_0^{-1}y$.

4) For every α $(0 < \alpha < 1)$ one can specify constants $C > 0$ and β $(0 < \beta < 1)$ such that

$$\sum_{q=k}^{\infty} \sum_{s=0}^{n-1} \frac{\left\|L_0^{-(q-k)}\right\|}{\left\|L_0^{-q}\right\|} \alpha^{qn+s} \leq C \frac{\beta^{kn}}{\left\|L_0^{-k}\right\|},$$

whatever positive integer k may be $(L_0^{-k} = (L_0^{-1})^k)$.

Denote by R the class of operators possessing the indicated properties (for different operators the number n may be different).

In the present note it is established that all operators of the class R for which the dimension of the subspace X_0 is the same are equivalent in a certain sense, explained below (see item 3). From this, in particular, follow the known results of M. K. Fage (^{1}) and Yu. N. Valitskii (^{2}), who considered similar questions for ordinary differential and integro-differential operators. The method we use is an abstract analogue of the method developed in (¹). The terminology used in (¹,²) is employed (L-basis, L-analyticity, etc.).

1. We shall call an L-basis a sequence of elements

$$f_0, f_1, \dots, f_m, \dots, \tag{1}$$

which is constructed in the following way. Its first n terms $f_0, f_1, \dots, \dots, f_{n-1}$ are the elements chosen in the subspace X_0 in constructing the operator L_0^{-1}; the remaining terms are defined recursively by the formula

$$f_m = L_0^{-1} f_{m-n} \quad (m = n,\ n+1, \dots). \tag{2}$$

Obviously,

$$\varphi_r(L^q f_m) = \begin{cases} 1 & \text{if } m = qn + r, \\ 0 & \text{if } m \neq qn + r \end{cases} \tag{3}$$

$$(q = 0, 1, 2, \dots; \quad r = 0, 1, \dots, n-1).$$

Theorem 1. Suppose that for the numerical sequence $\{a_m\}$ $(m = 0, 1, 2, \dots)$ the inequalities

$$|a_{qn+r}| \leq C \frac{\alpha^{qn+r}}{\|L_0^{-q}\| \, \|f_r\|}$$

$$(C > 0;\ 0 < \alpha < 1;\ q = 0, 1, 2, \dots;\ r = 0, 1, \dots, n-1) \tag{4}$$

hold. Then: a) the series

$$a_0 f_0 + a_1 f_1 + \dots + a_{mf} m + \dots = x \tag{5}$$

converges absolutely; b) $x \in \prod_{k=1}^{\infty} D(L^k)$; c) for every $k = 1, 2, \dots$

$$L^k x = \sum_{m=kn}^{\infty} a_m f_{m-kn}; \tag{6}$$

d) the series (6) converge absolutely, and moreover

$$\|L^k x\| \leq C_1 \frac{\beta^{kn}}{\|L_0^{-k}\|} \qquad (C_1 > 0;\ 0 < \beta < 1;\ k = 1, 2, \dots); \tag{7}$$

e) the formulas

$$a_{kn+r} = \varphi_r(L^k x) \qquad (k = 0, 1, 2, \dots;\ r = 0, 1, \dots, n-1) \tag{8}$$

hold.

Proof. The absolute convergence of the series (5) follows from the inequalities (4), and that of the series (6) from the inequalities (4) and property 4) of the operator L. Let y be the sum of the series (6). Then, by the continuity of the operator L_0^{-k},

$$L_0^{-k} y = \sum_{m=kn}^{\infty} a_m f_m = x - \sum_{s=0}^{kn-1} a_s f_s.$$

Hence $y = L^k x$. The inequalities (7) follow from property (4) of the operator L. The formulas (8) are obtained by computing $\varphi_r(L^k x)$ from the equalities (6), (3) and using the fact that the series (5) converges not only in the norm $\| \ \|$, but also in $\| \ \|_1$.

Corollary. If the series (5) with coefficients satisfying the conditions (4) has sum $x = 0$, then all its coefficients are equal to zero.

For every element $x \in \prod_{k=1}^{\infty} D(L^k)$ a series (5) can be constructed with coefficients computed by the formulas (8). The question of the relation between such series and the elements generating them is resolved by the following theorem.

Theorem 2. Suppose that for an element $x \in \prod_{k=1}^{\infty} D(L^k)$ the inequalities (7) hold. Then the coefficients of the series (5), computed by the formulas (8), satisfy inequalities of the type (4), and its sum is equal to x.

Proof. From the formulas (8) and the inequalities (7) it follows that

$$|a_{kn+s}| \le C_1 \left(\frac{\beta^{kn}}{\|L_0^{-k}\|} + \frac{\beta^{(k+1)n}}{\|L_0^{-(k+1)}\|} \right) \|\varphi_s\|.$$

Using property 4) of the operator L, we find constants $C_2 > 0$ and γ $(0 < \gamma < 1)$ such that

$$\frac{\beta^{(k+1)n}}{\|L_0^{-(k+1)n}\|} \le C_2 \frac{\gamma^{kn}}{\|L_0^{-1}\| \|L_0^{-k}\|}.$$

It follows from this that

$$|a_{kn+s}| \leq C_3 \frac{\delta^{kn}}{\|L_0^{-k}\|} \|\varphi_s\| \quad (0 < \delta = \max(\beta\gamma) < 1).$$

These inequalities are equivalent to the inequalities

$$|a_{kn+s}| \leq C_4 \frac{\delta^{kn+s}}{\|L_0^{-k}\|\|f_s\|} \quad (k = 0, 1, 2, \dots; \ s = 0, 1, \dots, n-1).$$

Let us prove the second assertion of the theorem. Let

$$x = a_0 f_0 + a_1 f_1 + \dots + a_{qn+s} f_{qn+s} + r_{qn+s}.$$

Then

$$L^q x = a_{qn} f_0 + a_{qn+1} f_1 + \dots + a_{qn+s} f_s + \rho_{qn+s},$$

where $\rho_{qn+s} = L^q r_{qn+s}$. For any $k = 0, 1, \dots, q-1$ and for arbitrary $p = 0, 1, \dots, n-1$ we have

$$\varphi_p(L^k r_{qn+s}) = \varphi_p(L^k x - a_{kn} f_0 - \dots - a_{qn+s} f_{(q-k)n+s}) = a_{kn+p} - a_{kn+p} = 0.$$

Therefore $r_{qn+s} = L_0^{-q} \rho_{qn+s}$. But

$$\|\rho_{qn+s}\| \leq \|L^q x\| + \sum_{r=0}^{s} \|a_{qn+r} f_r\| \leq C_5 \frac{\delta^{qn}}{\|L_0^{-q}\|}$$

$$(C_5 > 0, \ 0 < \delta < 1).$$

Hence $\|r_{qn+s}\| \leq C_6 \delta^{qn+s} \to 0$ as $qn+s \to \infty$. Applying Theorem 1, we see that if the conditions of Theorem 2 are satisfied, then the equalities (6) are valid.

What has been set forth above gives grounds, by analogy with (^{1}), to adopt the following:

Definition 1. An element

$$x \in \prod_{k=1}^{\infty} D(L^k)$$

is called ***L*-analytic** if the inequalities (7) are satisfied for it. The series (5), formed for the L-analytic element x by formulas (8), is called its ***L*-series**.

We note that the set $A_{L,f,\varphi}$ of L-analytic elements is determined not only by the operator L, but also by the choice of the first n elements of the L-basis and of the system of functionals biorthogonal to them.

2. Definition 2. A sequence x_k $(k = 1, 2, \ldots)$ of the set $A_{L,f,\varphi}$ will be called **regularly convergent** if: a) all sequences $L^q x_k$ $(q = 0, 1, 2, \ldots)$ converge in the norm $\| \ \|$ as $k \to \infty$, and b) there exist constants $C_1 > 0$ and α $(0 < \alpha < 1)$, independent of k, such that

$$\|L^q x_k\| \leq C_1 \frac{\alpha^{qn}}{\|L_0^{-q}\|} \quad (q = 1, 2, \ldots). \tag{7'}$$

Theorem 3. *If* $x \in A_{L,f,\varphi}$ $(k = 1, 2, \ldots)$ *and* $x_k \to x$ *as* $k \to \infty$ *regularly, then* $x \in A_{L,f,\varphi}$*, and moreover* $L^q x_k \to L^q x$ *as* $k \to \infty$ $(q = 1, 2, \ldots)$.

Proof. Put

$$x^{(q)} = \lim_{k\to\infty} L^q x_k \quad (q = 0, 1, 2, \ldots; \ x^{(0)} = x).$$

Since $L_0^{-q}(L^q x_k) \to L_0^{-q}(x^{(q)})$ as $k \to \infty$, we have

$$\begin{aligned} &x_k - \varphi_0(x_k) f_0 - \cdots - \varphi_{n-1}(x_k) f_{n-1} - \cdots \\ &\quad - \varphi_0(L^{q-1} x_k) f_{(q-1)n} - \cdots - \varphi_{n-1}(L^{q-1} x_k) f_{(q-1)n+n-1} \to L_0^{-q}(x^{(q)}). \end{aligned} \tag{9}$$

From the second condition of the theorem and the boundedness of the functionals $\{\varphi_s\}$ in the norm $\| \ \|_1$, it follows that the sequences $\varphi_s(L^q x_k)$ are fundamental. Let

$$a_{qn+s} = \lim_{k\to\infty} \varphi_s(L^q x_k) \quad (q = 0, 1, \ldots; \ s = 0, 1, \ldots, n-1).$$

Then from relation (9) we find that

$$x - a_0 f_0 - a_1 f_1 - \cdots - a_{n-1} f_{n-1} - \cdots - a_{(q-1)n} f_{(q-1)n} - \cdots$$

$$\cdots - a_{(q-1)n+n-1} f_{(q-1)n+n-1} = L_0^{-q}(x^{(q)}),$$

whence $L^q x = x^{(q)}$ $(q = 1, 2, \ldots)$. The inclusion $x \in A_{L,f,\varphi}$ follows from (7).

Remark 1. The numbers a_{qn+s} considered in the proof of Theorem 3 are, as is not hard to see, the coefficients of the L-series of the element x. For them inequalities of type (4) hold. The same inequalities, with constants not depending on k, hold also for the coefficients of the L-series of the elements x_k.

It is also easy to prove, using the estimate of the remainder of the L-series (see the proof of Theorem 2), that if for the coefficients of some sequence of L-series estimates of type (4) hold with constants not depending on k, and they converge to the coefficients of some L-series, then the sum of the latter is the limit (in the sense of regular convergence) of the sums of the sequence of L-series under consideration.

Remark 2. Theorem 3 shows that the operator L, considered on the set $A_{L,f,\varphi}$ with the topology determined by regular convergence, is continuous.

3. Let L and M be two operators of class R. Denote by n and ν, respectively, the dimensions of the subspaces of solutions of the equations $Lx = 0$ and $Mx = 0$; by $A_{L,f,\varphi}$ and $A_{M,g,\psi}$, respectively, the sets of L- and M-analytic elements (the systems of functionals $\{\varphi\}$ and $\{\psi\}$ being, generally speaking, different).

Let the operator T assign to the L-analytic element

$$x = \sum_{m=0}^{\infty} a_m f_m$$

the M-analytic element

$$y = \sum_{m=1}^{\infty} a_m g_m.$$

Such an operator gives an isomorphic (with preservation of regular convergence) mapping of $A_{L,f,\varphi}$ onto $A_{M,g,\psi}$.

Theorem 4. *If $\nu = n$, then the operators L and M are equivalent; namely, they are transformed into one another according to the rule $M = TLT^{-1}$.*

Indeed, in this case from the fact that $v = Tu$ it follows, by virtue of equalities (6), that $Mv = TLu$.

4. In the work of M. K. Fage (^{1}), functions analytic with respect to the operator

$$L = D^n + p_{n-1}(\xi)D^{n-1} + \cdots + p_0(\xi) \qquad \left(a < \xi < b, \quad D = \frac{d}{d\xi}\right)$$

were considered. In the work of Yu. N. Valitskii (^{2}), functions analytic with respect to the operator

$$\Lambda[y] = y^{(n)}(\xi) + \sum_{i=0}^{n-1} p_i(\xi) y^{(i)}(\xi) + \sum_{i=0}^{n-1} \int_{\xi_0}^{\xi} H_i(\xi, t) y^{(i)}(t)\, dt \qquad (\xi_0,\ \xi \in (a, b)).$$

The results of these authors on L- and Λ-analytic functions and on the equivalence of operators of the same order, pertaining to the case when $a \neq -\infty$, $b \neq +\infty$, fit within the framework of the abstract scheme set forth above if, taking as X the space of functions continuous on $[a, b]$, one puts $\varphi_s(y) = y^{(s)}(\xi_0)$ $(s = 0, 1, \dots, n-1;\ a < \xi_0 < b)$.

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2. Yu. N. Valitskii, *Dop. AN URSR*, No. 3, 237 (1959).

Note: Figure translations are in progress. See original paper for figures.

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