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Abstract

Full Text

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On Some Homogeneous Boundary-Value Problems for Differential Operators with Constant Coefficients

(Presented by Academician S. L. Sobolev on 2 III 1961)

The Cauchy problem for an equation of the form

$$\frac{\partial^2 Lu}{\partial t^2} + Mu = 0 \quad (1)$$

in the case when L is the Laplace operator in three variables and $M = \partial^2/\partial z^2$, was first investigated by S. L. Sobolev ⁽¹⁾, who obtained an explicit expression for the solution by means of the fundamental solutions he constructed. In the case when L is the Laplace operator in many variables, this problem was solved in the work of S. A. Galpern ⁽²⁾. As for the corresponding mixed problem, it was first studied in detail in the case of two independent variables in the works of R. A. Aleksandryan ^(3,4).

In the general case, the solution of the mixed problem by the Fourier method naturally leads to the following homogeneous boundary-value problem:

$$Mu - \lambda Lu = 0; \quad (2)$$

$$Gu|_{\Gamma} = 0, \quad (3)$$

which consists in finding those values of the numerical parameter λ for which there exist solutions $u_{\lambda}(x_1, \dots, x_n)$ of this problem such that $Gu_{\lambda} \neq 0$.

In ⁽⁵⁾ the case was considered in which equation (2) is the string equation, while the boundary operator G is a differential operator with constant coefficients of arbitrary order; using the results of R. A. Aleksandryan ⁽⁶⁾, the completeness of the corresponding system of eigenfunctions was proved.

We note that the method of constructing eigenfunctions given in ⁽⁵⁾ is connected with the form of the general solution of the string equation and is not applicable in the case of many independent variables.

Problem (2), (3), in the case when equation (2) is the wave equation with three variables in an arbitrary ellipsoid and G is the identity operator, was considered by R. T. Dzhenchev.

In the case when M and L are differential operators with constant coefficients of the form

$$M = \sum_{i,j=1}^n \alpha_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}, \quad L = \sum_{i,j=1}^n \beta_{i,j} \frac{\partial^2}{\partial x_i \partial x_j},$$

the quadratic form of the operator L is positive definite, G is the identity operator, and the equation is considered in some ellipsoid with center at the origin, we shall call problem (2), (3) problem C .

In the works ^(6,7,4) it was established that the polynomial eigenfunctions of problem C form a basis in the space of polynomials vanishing on the boundary.

In this note we consider problem C^* , which differs from problem C only in that G is not the identity operator, but a differential operator of arbitrary order.

Let $\overset{0}{W}_p^{(l)}(D)$ denote the closure of all finite and infinitely differentiable functions in D in the metric of the Sobolev space $W_p^{(l)}(D)$ ($p \geq 1, l \geq 0$).

Consider the following operators with constant coefficients:

$$A = \sum_{i_1+\dots+i_n=0}^N a_{i_1,\dots,i_n} \frac{\partial^{i_1+\dots+i_n}}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}, \quad B = \sum_{i=0}^m b_i \left(x_1 \frac{\partial}{\partial x_1} + \dots + x_n \frac{\partial}{\partial x_n} \right)^i,$$

where N and m are arbitrary natural numbers, and the operator $(x_1 \partial / \partial x_1 + \dots + x_n \partial / \partial x_n)^r$ is obtained by formal exponentiation.

It turns out that, with respect to the system of eigenfunctions generated by problem C^* , the following theorem holds:

Theorem. *If $G = AB$, $a_{0,\dots,0} = 1$, and if*

$$b_0 + kb_1 + \dots + k(k-1) \dots (k-m+1)b_m \neq 0$$

for every natural k , then the eigenfunctions of problem C^ form a complete system in a certain subspace $\widetilde{W}_p^{(l)}(D)$ of the space $W_p^{(l)}(D)$, which contains $\overset{0}{W}_p^{(l)}(D)$.*

Let \mathcal{P} be the space of all polynomials. We shall show that the operator G , considered only on the space \mathcal{P} , has an inverse G^{-1} defined on all of \mathcal{P} . For this it is enough to show that the operators A and B have this property.

The existence of A^{-1} follows from the fact that the equation

$$Ap = q(x_1, \dots, x_n)$$

for any q from \mathcal{P} has the solution

$$p = q - A_1 q + A_1^2 q - \dots + (-1)^m A_1^m q,$$

where m is the degree of the polynomial q , and

$$A_1 = \sum_{i_1 + \dots + i_n = 1}^N a_{i_1, \dots, i_n} \frac{\partial^{i_1 + \dots + i_n}}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}.$$

On the other hand, since $a_{0, \dots, 0} \neq 0$, the homogeneous equation $Ap = 0$ in the space \mathcal{P} has only the trivial solution.

Denote by \mathcal{P}_k the space of homogeneous polynomials of degree k . Introducing spherical coordinates, it is easy to show that if $p(x_1, \dots, x_n) \in \mathcal{P}_k$, then

$$Bp = \gamma_k p, \tag{4}$$

where $\gamma_k = b_0 + kb_1 + \dots + k(k-1) \dots (k-m+1)b_m$. Hence the existence follows of an inverse operator G^{-1} , defined on all of \mathcal{P} .

Let λ be an eigenvalue, and let $T_\lambda(x_1, \dots, x_n)$ be the corresponding polynomial eigenfunction of problem C.

We shall show that $P_\lambda = G^{-1}T_\lambda$ is an eigenfunction of problem C* corresponding to the same λ . We have $GP_\lambda = T_\lambda$, and since $T_\lambda|_\Gamma = 0$, it follows that P_λ satisfies the boundary condition of problem C*. It remains to show that P_λ satisfies the equation

$$Mu - \lambda Lu = 0. \tag{2*}$$

We have

$$(M - \lambda L)T_\lambda = (M - \lambda L)ABP_\lambda = A(M - \lambda L)BP_\lambda = 0,$$

whence it follows that

$$(M - \lambda L)BP_\lambda = 0. \tag{5}$$

Let m be the degree of the polynomial P_λ . We represent it in the form

$$P_\lambda = \sum_{k=0}^m P_{\lambda,k}, \quad \text{where } P_{\lambda,k} \in \mathcal{P}_k. \quad (6)$$

From (6), (4), and (5) we obtain

$$(M - \lambda L)BP_\lambda = (M - \lambda L)B \sum_{k=0}^m P_{\lambda,k} = (M - \lambda L) \sum_{k=0}^m \gamma_k P_{\lambda,k} = 0. \quad (7)$$

Taking into account that $\gamma_k \neq 0$ and that in equation (2*) only derivatives of one and the same order occur, from (7) we immediately conclude that P_λ satisfies equation (2*) and, consequently, is an eigenfunction of problem C^* .

Denote by \mathcal{P}_0 the space of polynomials vanishing on the boundary of the ellipsoid, and by \mathcal{P}_0^* the space of polynomials satisfying the boundary condition of problem C^* .

Let T_i be the set of all polynomial eigenfunctions of problem C (this collection of polynomials forms a basis in \mathcal{P}_0). The operators G and G^{-1} establish a one-to-one correspondence between \mathcal{P}_0 and \mathcal{P}_0^* . In particular, the set T_i from \mathcal{P}_0 corresponds in \mathcal{P}_0^* to the set of polynomial eigenfunctions $p_i^* = G^{-1}T_i$ of problem C^* , which, evidently, forms a basis in \mathcal{P}_0^* .

Consequently, if we denote by $\widetilde{W}_p^{(l)}(D)$ the completion of the set \mathcal{P}_0^* in the metric $W_p^{(l)}(D)$, then the eigenfunctions of problem C form a complete system in $\widetilde{W}_p^{(l)}(D)$. Obviously, $\widetilde{W}_p^{(l)}(D)$ belongs to $W_p^{(l)}(D)$, and it remains to show that it contains $W_p^{(l)}(D)$.

Let v belong to $W_p^{(l)}(D)$. For fixed ε there exists a finite, infinitely differentiable in \overline{D} function φ such that

$$\|v - \varphi\|_{W_p^{(l)}(D)} < \varepsilon.$$

On the other hand, there exists a sequence of polynomials Q_n which converges to φ in the metric $C^{(l)}(D)$, where $C^{(l)}(D)$ is the space of continuous functions continuously differentiable up to order l in \overline{D} , with norm

$$\|\varphi\|_{C^{(l)}(D)} = \max_{\substack{x \in \overline{D} \\ 0 \leq k \leq l}} |D^k \varphi|.$$

Let D' be some closed domain lying entirely in D and outside which the function φ is equal to zero. Construct the polynomials

$$R_n = [(g - 1)^s + 1]^\mu Q_n,$$

where $g(x_1, \dots, x_n) = 1$ is the equation of the ellipsoid; s is an odd number greater than $m + N$; μ is a positive integer. Taking into account the uniform boundedness of Q_n in the metric $C^{(l)}(D)$, and the fact that in D'

$$(g - 1)^s + 1 \leq \sigma < 1,$$

it is easy to see that, by choosing μ sufficiently large, one can make R_n arbitrarily small in the metric $C^{(l)}(D')$, independently of n . From the convergence of Q_n to φ it follows that Q_n tends to zero in the metric $C^{(l)}(D - D')$. Therefore, after choosing μ , one can choose n so large that the norms of $\varphi - Q_n$ and R_n in the metric $C^{(l)}(D)$ are less than ε . Comparing the preceding arguments, we conclude that, for the given v

and there exist polynomials Q and $R = [(g - 1)^s + 1]^\mu Q$ such that

$$\|v - (Q - R)\|_{W_p^{(l)}(D)} < 3\varepsilon.$$

At the same time, the multiplier $[(g - 1)^s + 1]^\mu$ is equal to one on the boundary, and all its derivatives up to order $s - 1$ are equal to zero on the boundary. Therefore, since $s - 1 \geq m + N$, it follows that $GR|_\Gamma = GQ|_\Gamma$, i.e. $(Q - R) \in \mathfrak{F}_0^*$.

Thus we have shown that if $v \in W_p^{(l)}(D)$, then it is the limit in the norm $W_p^{(l)}(D)$ of a sequence from \mathfrak{F}_0^* , i.e. $v \in \widetilde{W}_p^{(l)}(D)$. This completes the proof of the theorem.

Remark. The theorem remains true also in the case when the operator B is replaced by the operator

$$\overline{B} = \sum_{i=0}^m b_i \left[(x_1 - c_1) \frac{\partial}{\partial x_1} + \dots + (x_n - c_n) \frac{\partial}{\partial x_n} \right]^i.$$

The proof of this case is reduced to the preceding one by a change of variables. When D is the unit sphere, the boundary condition

$$\sum_{i=0}^m b_i \left(x_1 \frac{\partial}{\partial x_1} + \dots + x_n \frac{\partial}{\partial x_n} \right)^i u \Big|_\Gamma = 0$$

takes the form

$$\sum_{i=0}^m b_i \frac{\partial^i u}{\partial n^i} \Big|_\Gamma = 0,$$

where n is the outward normal.

In the case when the domain is a disk, taking A and B successively to be identical operators, and $l = 0$, we obtain the cases considered in (5).

Let us note that the theorem proved is also valid in the case of boundary conditions of the form

$$G = \sum_{i=1}^r A_1^{(i)} B_1^{(i)} A_2^{(i)} B_2^{(i)} \dots A_m^{(i)} B_m^{(i)}$$

under the following assumptions: the operators $A_j^{(1)}, B_j^{(1)}$ ($j = 1, 2, \dots, m$) satisfy the conditions of the theorem, while in each of the remaining summands the coefficient $a_{0, \dots, 0}$ of at least one of the operators A is equal to zero.

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Note: Figure translations are in progress. See original paper for figures.

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