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Abstract

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SOME THEOREMS ON THE ZEROS AND EXTREMAL PROPERTIES OF ENTIRE FUNCTIONS

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1°. In this note upper and lower estimates are given for entire functions of one class on the real axis, and new properties of the distribution of zeros are established. This makes it possible to substantially refine the extremal properties of entire functions.

2°. Theorem 1. Let $v(z)$ be the function obtained from the function $\sin \frac{\pi}{h} z$ by a shift of the zeros, more precisely:

$$v(z) = [z + h(1 - \delta(-1))] [z - h(1 - \delta(1))] \times \\ \times \prod_{\nu=2}^{\infty} \left(1 + \frac{z}{h(\nu - \delta(-\nu))}\right) e^{-z/\nu h} \prod_{\nu=2}^{\infty} \left(1 - \frac{z}{h(\nu - \delta(\nu))}\right) e^{z/\nu h}, \tag{1}$$

and let the shifts δ satisfy the conditions

$$|\delta(n)| < A < +\infty, \qquad |n(\delta(n+1) - \delta(n))| < B < +\infty. \tag{2}$$

Then along the real axis

$$v(x) = \varepsilon(x)(x - a')(a'' - x)\Phi_v(x) \exp \sum_{\nu=1}^{[x/h]} \frac{\delta(-\nu) + \delta(\nu) - 1}{\nu}, \tag{3}$$

where $\varepsilon(x) = \pm 1$; $a'(x), a''(x)$ are the zeros of the function $v(x)$ nearest to the point x; $\Phi_v(x)$ is bounded below and above, and

$$0 < \alpha(A, B) < |\Phi_v(x)| < \beta(A, B) < +\infty. \tag{3'}$$

Theorem 2. Let $\omega(z)$ be an entire function of class B with respect to the upper half-plane (see ([1])); $f(z)$ an entire function, and let the ratios $f(z) : \omega(z)$ and

$\overline{f(z)} : \omega(z)$ (see ([1])) be bounded for $|z| > R$, $\operatorname{Im} z \geq 0$. Let T be the closure of the image of the real axis of the z-plane under the mapping $t = \omega(z) : f(z)$. The point $t = 0$ belongs to one of the domains D complementary to the set T, and for every $t \in D$ the function $\omega_t(z) = \omega(z) - tf(z)$ belongs to class B and has in the upper half-plane as many zeros as the function $\omega(z)$.

3°. We give special cases of two theorems which will be formulated below in general form.

Theorem 3. Let $f(z)$ be a real entire function of exponential type $\sigma \leq \frac{\pi}{h}$, and let $L_f = \sup |f(x)| < \infty$, $-\infty < x < +\infty$. Let

the points ξ alternate with the zeros of the function $\cos \frac{\pi}{h}(z{-}\alpha)$, $\left|\cos \frac{\pi}{h}(\xi - \alpha)\right| > \varepsilon > 0$,

$$\xi_{-\nu} = \alpha + \frac{1}{2}h - (\nu - \theta_{-\nu})h, \qquad \xi_\nu = \alpha + \frac{1}{2}h + (\nu - \theta_\nu)h, \tag{4}$$

$$0 \leqslant \theta \leqslant 1; \qquad \nu = 1, 2, \dots,$$

and the conditions are satisfied:

$$|m(\theta_{m+1} - \theta_m)| < B, \qquad \sum_{\nu=1}^{N} \frac{\theta_{-\nu} + \theta_\nu - 1}{\nu} > c > -\infty. \tag{5}$$

If at all points ξ, $f(\xi) : \cos \frac{\pi}{h}(\xi - \alpha) \leqslant 1$, then all zeros of the function

$$\psi(z) = \cos \frac{\pi}{h}(z - \alpha) - f(z)$$

are real and alternate with the points ξ, provided only that $\psi(z)$ is not identically zero.

Two sequences strictly alternate if no coincidence of any points is allowed. Two sequences that are limits of strictly alternating ones will also be called alternating.

Theorem 4. If $f(z)$ is an entire real function of exponential type $\sigma \leqslant \frac{\pi}{h}$ and at adjacent points ξ takes values of different signs, then all zeros of the function $f(z)$ are real and alternate with the points ξ.

Remark. The function $f(z)$ may vanish at some of the points ξ. It is necessary that at all points ξ at which $f(\xi) \neq 0$, the quantities $f(\xi_\nu) : \cos \frac{\pi}{h}(\xi - \alpha)$ be of one sign.

From Theorem 3 there follows the following extremal property of the function $\cos z$.

Theorem 5. If an entire real function of exponential type $\sigma \leqslant \frac{\pi}{h}$ satisfies the conditions $f(\xi) = L\cos\frac{\pi}{h}(\xi-\alpha)$ and $f(\bar{\xi}) = L\cos\frac{\pi}{h}(\bar{\xi}-\alpha)$, $\operatorname{Im}\xi > 0$, or the conditions $f(x_1) = L\cos\frac{\pi}{h}(x_1-\alpha)$, $f(x_2) = L\cos\frac{\pi}{h}(x_2-\alpha)$, where x_1 and x_2 are real and are not separated by points ξ, then either

$$f(x) \equiv L\cos\frac{\pi}{h}(x-\alpha),$$

or at at least one point ξ

$$f(\xi) : L\cos\frac{\pi}{h}(\xi-\alpha) > 1.$$

Under the first type of condition, in the role of ξ one can always take points of maximum of

$$\left|\cos\frac{\pi}{h}(x-\alpha)\right|.$$

With such a choice, at least at one point ξ,

$$|f(\xi)| > L$$

and

$$\operatorname{sign} f(\xi) = \operatorname{sign}\cos\frac{\pi}{h}(\xi-\alpha).$$

4°. Let $u(z)$ be an entire real function of exponential type and $L_u < +\infty$. By $E(u)$ denote the set of points of maximum deviation of $u(x)$ from zero, i.e., the set of real points at which $|u(x)| = L_u$. By $Tsh(u)$ denote any maximal subset of the set $E(u)$ at whose points $u(x)$ takes values alternately of different signs. The points of the set $E(u)$ split the real axis into a finite or countable set of intervals j*. The quantity

$$I(u) = \sum_j \left[\frac{N(j)}{2}\right] + r,$$

where $N(j)$ is the number of zeros of the function $u(z)$ on the interval j, and $2r$ is the number of complex zeros of the function $u(z)$, is called the defect index of the function $u(z)$. The defect index, in a somewhat different formulation, was introduced in paper ([2]).

* If $E(u)$ is empty, then j consists of one interval $(-\infty, +\infty)$. If $Tsh(u)$ does not coincide with $E(u)$, then there exist several sets $Tsh(u)$.

Functions of finite defect index possess remarkable properties. In paper (2) it was proved that if the comparison family consists of entire real functions of exponential type not exceeding σ, taking prescribed values at $2m$ fixed points, then the extremal function of the family that deviates least from zero on the real axis is a function with defect index less than m. In the same paper it was shown that a function of type σ with defect index l has the form: $u(z) = L_u \cos\varphi(z)$, where $\varphi(z)$ is a hyperelliptic integral

$$\varphi(z) = \sigma \int_{x_0}^{z} \frac{H(z)k(z)\overline{k}(z)\,dz}{\sqrt{p(z)\overline{p}(z)Q(z)\overline{Q}(z)}} + a, \qquad H(z) = \prod_{\mu=1}^{2(l-l)} (z - x_\mu), \qquad k(z) = \prod_{\nu=1}^{l} (z - z_\nu),$$

$$p(z) = \prod_{\mu=1}^{2(l-l)} (z - \sigma_\mu), \qquad Q(z) = \prod_{\nu=1}^{l} (z - \alpha_\nu)(z - \beta_\nu),$$

$$0 \le l \le I, \qquad \operatorname{Im} x_\mu = \operatorname{Im} x_0 = 0, \qquad \operatorname{Im} z_\nu > 0, \qquad \operatorname{Im} \alpha_\nu > 0, \qquad \operatorname{Im} \beta_\nu > 0.$$

For example, the only functions with zero defect index are functions of the form $L\cos(\sigma z + a)$. It is easy to see that the zeros of a function with finite defect index have the following asymptotics:

$$a_n \simeq \frac{n\pi}{\sigma} + \gamma + \frac{c}{n} + O\left(\frac{1}{n^2}\right). \tag{6}$$

5°. Between every two neighboring points b', b'' of the set $Tsh(u)$ there is an odd number of zeros of the function $u(z)$ (the number of intervals where this number is greater than 1 does not exceed $I(u)$). We discard the open interval with endpoints at the leftmost and rightmost zeros of the function $u(z)$ in (b', b''). We denote the remaining system of intervals by $\gamma_{-\nu} = (a_{-\nu}, a_{-\nu+1})$ and $\gamma_\nu = (a_{\nu-1}, a_\nu)$, where a_ν are the zeros of $u(z)$. In each interval we choose a point

$$\xi_{-\nu} = a_{-\nu} + \theta_{-\nu}(a_{-\nu+1} - a_{-\nu}), \qquad \xi_\nu = a_\nu - \theta_\nu(a_\nu - a_{\nu-1}),$$

$$0 \le \theta \le 1, \qquad \nu = 1, 2, \dots, \tag{7}$$

and define the function

$$v(z,u) = Q(z)e^{kz}(z-\xi_{-1})(z-\xi_1)\prod_{\nu=2}^{\infty}\left(1-\frac{z}{\xi_{-\nu}}\right)e^{z/a_{-\nu}}\prod_{\nu=2}^{\infty}\left(1-\frac{z}{\xi_{\nu}}\right)e^{z/a_{\nu}}, \quad (8)$$

where $Q(z)$ is an arbitrary real polynomial of degree $2I(u)$, having no real zeros. The constant k is chosen so that (see (2), p. 515)

$$\lim_{z\to\infty} e^{\eta|z|}\frac{\sqrt{L_u^2 - u^2(z)}}{v(z,u)} = \begin{cases} \infty & \text{for } \eta > 0, \\ 0 & \text{for } \eta < 0. \end{cases}$$

By virtue of the asymptotics (6), the function $v(x,u)$ satisfies estimate (3).

Theorem 6. Let $f(z)$ be an entire real function of exponential type $\sigma_1 \leq \sigma$, $L_f < +\infty$. If at the points of the sequence (7) $f(\xi) : u(\xi) \leq 1, \ |u(\xi)| > \varepsilon > 0$ and

$$|m(\theta_{m+1} - \theta_m)| < B < \infty, \qquad \sum_{\nu=1}^{N}\frac{\theta_{-\nu} + \theta_{\nu} - 1}{\nu} > c > -\infty, \quad (9)$$

then all zeros of the difference $\psi(z) = u(z) - f(z)$, with the possible exception of $2I(u)$ zeros, are real and alternate with the points ξ.

Theorem 7. If at neighboring points ξ the function $f(x)$ takes values of different signs, or, more generally, at all points ξ for which $f(\xi) \neq 0$ the ratio $f(\xi) : u(\xi)$ has one and the same sign, then all roots of the function $f(z)$,

except, perhaps, for $2I(u)$ roots, are real and alternate with the points ξ.

6°. Theorem 8. Let $\psi(z)$ be a real entire function of at most exponential type, and let several groups of consecutive derivatives at one or several real points be equal to zero,

$$\psi^{(k_1)}(x_1) = \dots = \psi^{(k_1+r_1-1)}(x_1) = 0, \dots, \quad \psi^{(k_\mu)}(x_\mu) = \dots = \psi^{(k_\mu+r_\mu-1)}(x_\mu) = 0. \quad (10)$$

Suppose, moreover, that the function $\psi(z)$, or some derivatives of the function $\psi(z)$, have zeros at $2j$ complex points $\zeta_1, \bar{\zeta}_1, \dots, \zeta_j, \bar{\zeta}_j$.

If $k_1 = k_2 = \dots = k_\nu = 0$, while the remaining $\mu - \nu$ quantities k are greater than 0, then $\psi(z)$ has at least ν real roots of multiplicities $r_1, \dots, r_\nu$ and at least

$$j + \left[\frac{r_{\nu+1}}{2}\right] + \dots + \left[\frac{r_\mu}{2}\right]$$

pairs of complex roots. The points $x_1, x_2, \dots, x_\mu$ may coincide.

Theorem 9. Let the comparison family consist of real entire functions of exponential type not exceeding σ, satisfying the following conditions: 1) the functions, or their derivatives of certain orders, take prescribed values at $2j$ complex points $\zeta_1, \bar{\zeta}_1, \ldots, \zeta_j, \bar{\zeta}_j$; 2) the values of consecutive derivatives of orders $k_1, \ldots, k_1 + 2l_1 - 1, \ldots, k_\mu, \ldots, k_\mu + 2l_\mu - 1$ at real points $x_1, \ldots, x_\mu$ have prescribed values.

If the function $u(z)$ with defect index $I(u) < j + l_1 + \ldots + l_\mu$ belongs to the comparison family, then $u(z)$ is an extremal function. Moreover, there always exists a function of type σ with index $I(u) < j + l_1 + \ldots + l_\mu$ belonging to the comparison family. If all $k > 0$, then it is the unique extremal function. If $k_1 = k_2 = \ldots = k_\nu = 0$, then the uniqueness condition is that the nodes $x_1, \ldots, x_\nu$ do not belong to the set $E(u)$.

Theorem 10. Let $I(u) < j + l_1 + \ldots + l_\mu$, and let the function $u(z)$ be the unique solution of the extremal problem. Let ξ be the points from (7) and suppose condition (9) is satisfied. If $f(z)$ is any function of the comparison family, distinct from $u(z)$, $L_f < \infty$, then at least at one point ξ

$$f(\xi) : u(\xi) > 1.$$

As the set of points ξ one may always take the set $Tsh\, u(z)$.

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2. N. N. Meiman, Tr. Moscow Math. Soc., **9**, 507 (1960).

Note: Figure translations are in progress. See original paper for figures.