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Abstract

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SYMMETRIC MOTIONS OF AN IDEAL FLUID FROM THE STATE OF ITS ROTATION AS A RIGID BODY*

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The first investigations of motions of the type under consideration go back to the works (^{1,2}). The present work is based on the following system of linearized equations, characterizing symmetric motions of an ideal incompressible fluid of density ρ, differing little from the rotation of the fluid as a rigid body about the x-axis with angular velocity ω:

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial x}; \qquad \frac{\partial v_r}{\partial t} - 2\omega v'_\theta = -\frac{1}{\rho}\frac{\partial p'}{\partial r};$$

$$\frac{\partial v'_\theta}{\partial t} + 2\omega v_r = 0; \qquad \frac{\partial (r v_x)}{\partial x} + \frac{\partial (r v_r)}{\partial r} = 0. \tag{1}$$

Here x, r, θ are cylindrical coordinates; t is time; $v'_\theta = v_\theta - \omega r$, $p' = p - \frac{1}{2}\rho\omega^2 r^2$; v_x, v_r, v_θ are the projections of the velocity vector on the coordinate lines; p is the pressure; v_x, v_r, v_θ, p depend only on x, r, t and do not depend on θ. System (1) is taken as the basis in works ($^{3-6}$). Integrating the third equation of system (1), with fixed x, r, with respect to t, we obtain

$$v'_\theta = -2\omega \int_0^t v_r \, dt = -2\omega r'.$$

The quantity r', small together with v'_θ, characterizes the radial deviation of fixed particles from their initial position. In this connection, solutions of system (1) are suitable only for that range of t in which the relative radial displacements r'/r of fixed particles are small. The fourth equation of system (1) makes it possible to introduce the "stream function" ψ, defined by the equalities

$$v_x = \frac{1}{r}\frac{\partial \psi}{\partial r}, \qquad v_r = -\frac{1}{r}\frac{\partial \psi}{\partial x}. \tag{2}$$

By virtue of the first three equations of system (1), the function ψ satisfies the equation

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{1}{r}\frac{\partial^2\psi}{\partial x^2}\right]+4\omega^2\frac{1}{r}\frac{\partial^2\psi}{\partial x^2}=0. \tag{3}$$

Equation (3) admits particular solutions of the form

$$\psi=\omega r_0^3\beta e^{2k\omega t}\Psi(X,R);\qquad x=r_0X,\quad r=r_0R,\quad \beta=\text{const.} \tag{4}$$

Here $\Psi(X,R)$ is a real function; k is a real or purely imaginary constant; r_0 is a characteristic constant with the dimension of length. The function $\Psi(X,R)$ satisfies the equation

$$(1+k^2)\frac{1}{R}\frac{\partial^2\Psi}{\partial X^2}+k^2\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right)=0. \tag{5}$$

Introduce, instead of X, the variable

$$X_1=|k|X/\sqrt{|1+k^2|}. \tag{6}$$

* The initial results of this work were reported by the author at the First All-Union Congress on Theoretical and Applied Mechanics, 13 I 1960.

In this case equation (5) takes the form

$$\frac{1}{R}\frac{\partial^2\Psi}{\partial X_1^2}+\frac{k^2(1+k^2)}{|k^2|\,|1+k^2|}\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right)=0. \tag{7}$$

Equation (7) is the condition for the existence of a function $\Phi(X_1,R)$, defined by the relations

$$\frac{\partial\Phi}{\partial X_1}=\frac{1}{R}\frac{\partial\Psi}{\partial R},\qquad \frac{\partial\Phi}{\partial R}=-\frac{k^2}{|k^2|}\frac{|1+k^2|}{1+k^2}\frac{1}{R}\frac{\partial\Psi}{\partial X_1}. \tag{8}$$

The function Φ satisfies the equation

$$R\frac{\partial^2\Phi}{\partial X_1^2}+\frac{k^2(1+k^2)}{|k^2|\,|1+k^2|}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right)=0. \tag{9}$$

In the case of purely imaginary $k=ik_1$, in equality (4) one must take the real and imaginary parts:

$$\psi=\omega r_0^3\Psi(X,R)\cos 2k_1\omega t, \tag{10}$$

$$\psi = \omega r_0^3 \Psi(X, R) \sin 2k_1 \omega t. \tag{11}$$

Let us note that the boundary condition $d\psi = 0$ for solutions of equation (3), characterizing the condition of non-flow of the liquid through a boundary invariant in time, obviously passes, for solutions of the form (4), (10), (11), into the boundary condition $d\Psi(X_1, R) = 0$ on the boundary of the plane X_1, R, transformed by virtue of the relations $x = r_0 X,\ r = r_0 R$ and relation (6). For real $k > 0$, relations (6), (7), (8), (9) take the form

$$X_1 = \frac{kX}{\sqrt{1+k^2}} = \frac{k}{\sqrt{1+k^2}} \frac{x}{r_0}; \tag{12}$$

$$\frac{1}{R} \frac{\partial^2 \Psi}{\partial X_1^2} + \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \Psi}{\partial R} \right) = 0; \tag{13}$$

$$\frac{\partial \Phi}{\partial X_1} = \frac{1}{R} \frac{\partial \Psi}{\partial R}, \qquad \frac{\partial \Phi}{\partial R} = -\frac{1}{R} \frac{\partial \Psi}{\partial X_1}; \tag{14}$$

$$R \frac{\partial^2 \Phi}{\partial X_1^2} + \frac{\partial}{\partial R} \left(R \frac{\partial \Phi}{\partial R} \right) = 0. \tag{15}$$

These equations coincide with the equations of potential axisymmetric motion of an incompressible fluid with velocity potential $\Phi(X_1, R)$ and stream function $\Psi(X_1, R)$. Equation (15) for Φ is the Laplace equation for the axisymmetric case, i.e. an equation of elliptic type. Comparing relations (1), (2), (12), (14) and requiring fulfillment of the identities $\psi \equiv 0,\ p' \equiv 0$ at $t = -\infty$, we obtain for solutions of type (4) with $k > 0$

$$p' = -2\sqrt{1+k^2} \rho \left(\omega r_0\right)^2 \beta \Phi(X_1, R) e^{2k\omega t}. \tag{16}$$

For imaginary $k = ik_1$ ($k_1 > 0$), two fundamentally different cases are possible. For $k_1 > 1$, relation (6) takes the form:

$$X_1 = \frac{k_1 X}{\sqrt{k_1^2 - 1}} = \frac{k_1}{\sqrt{k_1^2 - 1}} \frac{x}{r_0}, \tag{17}$$

and relations (7), (8), (9) are again reduced to relations (13), (14), (15); the problem is again reduced to the study of potential motions. Using relations (1), (2), (14), (17), we obtain for p' corresponding to (10): $p' = -2\sqrt{k_1^2 - 1} \rho (\omega r_0)^2 \beta \Phi(X_1, R) \sin 2k_1 \omega t$, and for p' corresponding to (11): $p' = -2\sqrt{k_1^2 - 1} \rho (\omega r_0)^2 \beta \Phi(X_1, R) \cos 2k_1 \omega t$.

For imaginary $k = ik_1$ and $0 < k_1 < 1$, relations (6), (7), (8), (9) take the form:

$$X_1 = \frac{k_1 X}{\sqrt{1-k_1^2}} = \frac{k_1}{\sqrt{1-k_1^2}}\frac{x}{r_0}; \qquad \frac{1}{R}\frac{\partial^2\Psi}{\partial X_1^2} - \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right) = 0; \tag{18a}$$

$$\frac{\partial\Phi}{\partial X_1} = \frac{1}{R}\frac{\partial\Psi}{\partial R}, \qquad \frac{\partial\Phi}{\partial R} = \frac{1}{R}\frac{\partial\Psi}{\partial X_1}; \qquad R\frac{\partial^2\Phi}{\partial X_1^2} - \frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) = 0. \tag{18 }$$

The second and fourth equations (18) are already equations of hyperbolic type, while the fourth equation for Φ is the equation of cylindrical waves. Similarly to the preceding case, for p' in the case of equality (10) we obtain: $p' = -2\sqrt{1-k_1^2}\rho(\omega r_0)^2\beta\Phi(X_1, R)\sin 2k_1\omega t$, and in the case of equality (11): $p' = -2\sqrt{1-k_1^2}\rho(\omega r_0)^2\beta\Phi(X_1, R)\cos 2k_1\omega t$.

Let us consider some problems reducible to the solution of equations of elliptic type. Suppose that in an unbounded fluid, initially rotating like a rigid body, in the direction of the negative x-axis there moves an axisymmetric body, symmetrically situated with respect to the x-axis, with maximum radial size r_0; the equations of its generators in the coordinate system fixed to the body are given in the form $x = r_0 f_1(r/r_0)$, $x = -r_0 f_2(r/r_0)$. Let at $t = -\infty$ $v_x = v_r = 0$, $v_\theta = \omega r$, and let the velocity of motion of the body vary according to the law

$$V_\infty(t) = \beta\omega r_0 e^{2k\omega t} \qquad (\beta = \text{const},\ k > 0). \tag{19}$$

In the coordinate system x, r fixed to the body, we obtain the following boundary-value problem for solutions $\Psi(x, r, t)$ of equation (3):

$$\psi(x, r, -\infty) \equiv 0, \qquad \psi(x, r, t) = 0 \quad \text{for } x = r_0 f_1(r/r_0),\ x = -r_0 f_2(r/r_0);$$

$$\text{for } \sqrt{x^2 + r^2} \to \infty \qquad \frac{1}{r}\frac{\partial\psi}{\partial r} \to V_\infty(t) = \beta\omega r_0 e^{2k\omega t}. \tag{20}$$

The solution of the problem will be found in the form (4) and will satisfy equations (12), (13), (14), (15) under the boundary conditions:

$$\Psi(X_1, R) = 0 \quad \text{for } X_1 = \frac{k}{\sqrt{1+k^2}} f_1(R), \quad X_1 = -\frac{k}{\sqrt{1+k^2}} f_2(R); \tag{21}$$

$$\frac{1}{R}\frac{\partial\Psi(X_1, R)}{\partial R} = \frac{\partial\Phi(X_1, R)}{\partial R} \to 1 \quad \text{for } \sqrt{X_1^2 + R^2} \to \infty. \tag{22}$$

Thus a problem is obtained of finding the “fictitious”potential flow around a body specified by equations (21), by a translational stream with velocity at infinity

$\partial\Phi/\partial X_1 = 1$. In this case the pressures must be calculated by formula (16), but with the replacement of the function $\Phi(X_1, R)$ by the function $\Phi_1(X_1, R) = \Phi(X_1, R) - X_1$; the latter replacement effects a return to the fixed coordinate system. Integrating, over the body, the projection of the pressure forces on the positive direction of the x-axis, we obtain for the drag force D the expression

$$D = 2\sqrt{1+k^2}\,\beta\omega^2 r_0 e^{2k\omega t} M(k) = \frac{\sqrt{1+k^2}}{k} M(k) \frac{dV_\infty}{dt}; \tag{23}$$

$$M(k) = 2\pi\rho r_0^3 \int_0^1 \left\{ \Phi_1\left[\frac{k}{\sqrt{1+k^2}} f_1(R), R\right] - \Phi_1\left[-\frac{k}{\sqrt{1+k^2}} f_2(R), R\right]\right\} R\,dR. \tag{24}$$

Obviously, $M(k)$ is the "added mass" in the usual hydrodynamic sense of this term (7) for displacements along the x-axis of a body whose generators have the equations $x = \dfrac{k}{\sqrt{1+k^2}} r_0 f_1\left(\dfrac{r}{r_0}\right)$,

$$x = -\frac{k}{\sqrt{1+k^2}} r_0 f_2\left(\frac{r}{r_0}\right),$$

i.e., of the original body compressed in the direction of the x-axis. The characteristic time scale of the motion of the body may be taken as the time interval $T_1 = 1/k\omega$ over which the quantity V_∞ in the dependence (19) changes by a factor e^2. The time of rotation through 1 radian at angular velocity ω is $T_0 = 1/\omega$. Thus, $k = T_0/T_1$. As $k \to \infty$, equation (23) gives $D = M(\infty) dV_\infty/dt$. Obviously, $\lim_{k\to\infty} M(k) = M(\infty)$ is the "added mass" of the undeformed body. For small $k \to 0$ the boundary conditions (21), (22) pass into the conditions of transverse potential flow-around ···

···by a potential "fictitious" flow of a circular disk. Obviously, for a body of any shape, expression (24) for $M(k)$ as $k \to 0$ tends to the "added mass" $M(0)$ of a circular disk of radius r_0, which, as is known (7), is equal to $M(0) = \frac{8}{3}\rho r_0^3$. In connection with this, as $k \to 0$ the principal term for the resistance expressed by formula (26) will be:

$$D = \frac{16}{3}\beta\rho\omega^2 r_0^4 e^{2k\omega t} = \frac{1}{k} M(0) \frac{dV_\infty}{dt} = \frac{1}{k}\cdot\frac{8}{3}\rho r_0^3 \frac{dV_\infty}{dt} = \frac{16}{3}\rho\omega r_0^3 V_\infty(t). \tag{25}$$

Formulas (25) show that in the expression for D in terms of the acceleration dV_∞/dt there stands the added mass $M(0)$ of the disk, divided by k, and, consequently, for small k the effective "added mass" becomes very large. Obviously, the solutions of the boundary-value problem characterized by conditions (20), taken for various combinations of β and k, may be summed. This makes it possible to solve the problem for arbitrary laws $V_\infty(t)$, specified in the form of integral representations of the form

$$V_\infty(t) = \omega r_0 \int_0^{k_0} \beta(k) e^{2k\omega t}\, dk, \qquad 0 < k_0 \ll \infty, \tag{26}$$

where $\beta(k)$ is an arbitrary function ensuring convergence of the integral and the necessary smallness of the quantity $V_\infty(t)/\omega r_0$. In this case, to obtain the resistance force it is necessary to integrate relation (23)

$$D = 2\omega^2 r_0 \int_0^{k_0} \beta(k) \sqrt{1+k^2}\, e^{2k\omega t} M(k)\, dk. \tag{27}$$

If in the integral representation (26) $k_0 \ll 1$, $k_0 \to 0$, then (27), using the equalities $(M(0) = \frac{8}{3}\rho r_0^3)$ and (26), takes the form

$$D = \frac{16}{3}\rho\omega^2 r_0^4 \int_0^{k_0} \beta(k) e^{2k\omega t}\, dk = \frac{16}{3}\rho\omega r_0^3 V_\infty(t). \tag{28}$$

The integral representation (26) with $k_0 \ll 1$ is in any case possible for arbitrary smooth motions of a body from a state of rest, whose duration in time is considerably greater than the time of rotation through 1 radian with angular velocity ω. Formula (28) shows that for such motions there holds a quite original law of resistance: the resistance depends only on the maximum radial dimension of the body and on its velocity at the given instant of time.

In papers ([5],[6]), by a completely different method, the particular problem was considered of the motion for $t > 0$ with constant velocity $V_0 = \text{const}$ of a sphere of radius r_0 in a liquid which at $t = 0$ was rotating with angular velocity ω. For the resistance D the asymptotic expression was obtained: $\lim_{t\to\infty} D = \frac{16}{3}\rho\omega r_0^3 V_0$. This expression is in agreement with the general law expressed by formula (28).

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1. J. Proudman, Proc. Roy. Soc., **92**, 408 (1916).

2. G. J. Taylor, Proc. 1-st Int. Congr. Appl. Mech., 1924, p. 99.

3. H. Görtler, Zs. angew. Math. u. Mech., **24**, Nos. 5 and 6 (1944).

4. G. W. Morgan, Proc. Roy. Soc., A, **206**, 108 (1951).

5. K. Stewartson, Proc. Cambr. Phil. Soc., **48**, 168 (1952).

6. G. W. Morgan, Proc. Cambr. Phil. Soc., **49**, 362 (1953).

7. N. E. Kochin, I. A. Kibel, N. V. Roze, *Theoretical Hydromechanics*, **1**, 1948.

Note: Figure translations are in progress. See original paper for figures.