A TWO-POINT BOUNDARY VALUE PROBLEM FOR A NONLINEAR DIFFERENTIAL EQUATION OF SECOND ORDER AND THEOREMS ON INTERMEDIATE VALUES

For the equation

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

G. A. KAMENSKY

A TWO-POINT BOUNDARY VALUE PROBLEM FOR A NONLINEAR DIFFERENTIAL EQUATION OF SECOND ORDER AND THEOREMS ON INTERMEDIATE VALUES

(Presented by Academician I. G. Petrovsky, March 18, 1961)

For the equation

$$y'' = F(x, y, y') \tag{1}$$

the boundary value problem

$$y(x_0) = y_0, \qquad y(x_1) = y_1 \tag{2}$$

is considered.

It is assumed concerning the function $F(x, y, v)$ that it is continuous in the domain $D\{x_0 \leq x \leq x_1;\ -\infty < y, v < +\infty\}$ and that certain conditions are satisfied which ensure the uniqueness of the solution of the initial value problem and the continuous dependence of solutions on the initial conditions.

There exist theorems establishing the solvability of the boundary value problem (1)–(2) for arbitrary y_0 and y_1 and for an interval length (x_0, x_1) not exceeding a certain quantity d, depending on the properties of the function $F(x, y, v)$.

Theorems 1 and 2 below make it possible, under certain conditions, using the solvability of the boundary value problem (1)–(2) for arbitrary y_0 and y_1 and for $x_1 - x_0 \leq d$, to obtain conditions for the solvability of the boundary value problem (1)–(2) for $x_1 - x_0 \leq 2d$ and for arbitrary y_0 and y_1.

Let $y(x, \alpha)$ be the solution of equation (1) with initial conditions $y(x_0, \alpha) = y_0,\ y'(x_0, \alpha) = \alpha;\ y(x, \beta)$ the solution of equation (1) with initial conditions $y(x_1, \beta) = y_1,\ y'(x_1, \beta) = \beta$. Denote $x_2 = \frac{1}{2}(x_0 + x_1)$, $\varphi(\alpha) = y(x_2, \alpha)$, $\psi(\beta) = y(x_2, \beta)$, $f(\alpha, \beta) = y'(x_2, \alpha) - y'(x_2, \beta)$.

We shall say that, for the function $\varphi(\alpha)$, condition (E) is fulfilled on the segment $[\alpha_0, \alpha_1]$ if $\varphi(\alpha_0) < \varphi(\alpha) < \varphi(\alpha_1)$ (or $\varphi(\alpha_0) > \varphi(\alpha) > \varphi(\alpha_1)$) for $\alpha_0 < \alpha < \alpha_1$.

Theorem 1. *Suppose:*

1) *The boundary value problem (1)–(2) is solvable for* $x_1 - x_0 \leq d$ *and for arbitrary* y_0 *and* y_1.

2) *There exist numbers* $t_0, t_1, \alpha_0, \alpha_1, \beta_0, \beta_1$ *such that* $\varphi(\alpha_0) = \psi(\beta_0) = t_0$, $\varphi(\alpha_1) = \psi(\beta_1) = t_1$; *for the functions* $\varphi(\alpha)$ *and* $\psi(\beta)$, *condition (E) is fulfilled on the segments* $[\alpha_0, \alpha_1]$ *and* $[\beta_0, \beta_1]$, *and the values* $f(\alpha_0, \beta_0)$ *and* $f(\alpha_1, \beta_1)$ *have opposite signs.*

Then the boundary value problem (1)–(2) is solvable for $x_1 - x_0 \leq 2d$ *and for arbitrary* y_0 *and* y_1.

Theorem 2. *Suppose:*

1) *The boundary value problem (1)–(2) is solvable for* $x_1 - x_0 \leq d$ *and for arbitrary* y_0 *and* y_1.

2) *There exist numbers* $t_0, t_1, \alpha_0, \alpha_1, \beta_0, \beta_1$ *such that* $\varphi(\alpha_0) = \psi(\beta_0) = t_0$, $\varphi(\alpha_1) = \psi(\beta_1) = t_1$; *the functions* $\varphi(\alpha)$ *and* $\psi(\beta)$ *are defined on the segments* $[\alpha_0, \alpha_1]$ *and* $[\beta_0, \beta_1]$, *and for any* α', α'' *from* $[\alpha_0, \alpha_1]$ *and* β', β'' *from* $[\beta_0, \beta_1]$ *satisfying the relations* $\varphi(\alpha') = \psi(\beta') = t_0$, $\varphi(\alpha'') = \psi(\beta'') = t_1$, *the values* $f(\alpha', \beta')$ *and* $f(\alpha'', \beta'')$ *have opposite signs.*

Then the boundary value problem (1)–(2) is solvable for $x_1 - x_0 \leq 2d$ *and for arbitrary* y_0 *and* y_1.

Using theorems on the solvability of the boundary-value problem (1)–(2) for $x_1 - x_0 \leq d'$ and some sufficient conditions for condition 2) in the formulation of Theorem 1 or Theorem 2 to hold, one can, by applying Theorem 1 or Theorem 2 successively the required number of times, obtain sufficient conditions for the solvability of the boundary-value problem (1)–(2) for arbitrary x_0, y_0, x_1, y_1.

Condition 2) will hold if (1) is satisfied and one of the following conditions holds:

a) The partial derivatives F_y and F_v exist in the domain D, and there exists an $m > 0$ such that $F_y \geq m$ in the domain D.

b) The function $F(x, y, v)$ is bounded in the domain D.

Condition 2) also holds if a combination of one of the conditions c) or d) and condition e) is satisfied:

c) The partial derivative F_{vvv} exists in the domain D, and there exists an $m > 0$ such that $F_{vvv} \leq -m$ in D.

d) There exist continuous positive functions $A(x, y)$ and $B(x, y)$ such that

$$|F(x, y, v)| \leq A(x, y)v^2 + B(x, y)$$

in the domain D.

e) There exists a number $L > 0$ such that $F(x, y, 0) > 0$ for $y > L$ and $F(x, y, 0) < 0$ for $y < -L$.

A sufficient condition for the boundary-value problem (1)–(2) to be solvable when the length of the interval (x_0, x_1) is less than some $d > 0$ is Opial' s theorem (^{2}). Using Opial' s theorem and sufficient conditions for condition 2) to hold, we obtain the following theorem.

Theorem 3. Let the function $F(x, y, v)$ be continuous in the domain D, let the derivatives F_y and F_v be continuous in D, and suppose there exist constants $M > 0$ and $K > 0$ such that

$$|F_y(x, y, v)| < M, \qquad F_v(x, y, v) < K$$

in the domain D, and suppose that condition a), or the combination of conditions c) and e), is satisfied:

a) There exists an $m > 0$ such that $F_y \geq m$ in the domain D.

b) There exists an $m > 0$ such that $F_{vvv} \leq -m$ in the domain D.

c) There exists an $L > 0$ such that $F(x, y, 0) > 0$ for $y > L$ and $F(x, y, 0) < 0$ for $y < -L$.

Then the boundary-value problem (1)–(2) is solvable for arbitrary x_0, x_1, y_0, y_1.

Another sufficient condition for the boundary-value problem (1)–(2) to be solvable when the length of the interval (x_0, x_1) is less than some d is

Theorem 4. Suppose there exist positive constants A and B such that

$$|F(x, y, v)| < Av^2 + B$$

in the domain D.

Then, whatever y_0 and y_1 may be, there exists a $d > 0$ such that the boundary-value problem (1)–(2) is solvable for $x_1 - x_0 \leq d$.

From Theorem 2, Theorem 4, and the sufficient conditions given above for condition 2) in the formulation of Theorem 2 to hold, there follow S. N. Bernstein' s theorem (^{1}), Z. F. Surikova' s theorem (^{4}), and the Scorza–Dragoni theorem (^{3}).

The proof of Theorems 1 and 2 is based on the following intermediate-value theorems for the superposition of a continuous function of two variables and two functions of one variable inverse to continuous ones.

Let functions $\varphi(\alpha)$ and $\psi(\beta)$ be given on $[\alpha_0, \alpha_1]$ and $[\beta_0, \beta_1]$. Denote by $G(\varphi, \psi)$ the set of those points of the rectangle $B\{\alpha_0 \leq \alpha \leq \alpha_1;\ \beta_0 \leq \beta \leq \beta_1\}$ for which $\varphi(\alpha) = \psi(\beta)$.

Theorem 5. Let the function $f(\alpha, \beta)$ be continuous on the rectangle B, let the functions $\varphi(\alpha)$ and $\psi(\beta)$ be continuous on the intervals $[\alpha_0, \alpha_1]$ and $[\beta_0, \beta_1]$, let condition (E) hold for $\varphi(\alpha)$ and $\psi(\beta)$ on these intervals, and let $\varphi(\alpha_0) = \psi(\beta_0)$, $\varphi(\alpha_1) = \psi(\beta_1)$.

Then there exists a component of the set $G(\varphi, \psi)$ connecting the points (α_0, β_0) and (α_1, β_1).

There are examples in which the hypotheses of Theorem 5 are satisfied, but the set $G(\varphi, \psi)$ turns out to be disconnected.

It follows from Theorem 5 that if the hypotheses of the theorem are satisfied, $f(\alpha_0, \beta_0) = c_0$ and $f(\alpha_1, \beta_1) = c_1$, then, whatever c lying between c_0 and c_1 may be, there exists a point (α, β) of the set $G(\varphi, \psi)$ such that $f(\alpha, \beta) = c$.

Let us note that failure of condition (E) may lead to the nonexistence of a component of the set $G(\varphi, \psi)$ connecting the points (α_0, β_0) and (α_1, β_1).

Theorem 6. Let a continuous function of two variables $f(\alpha, \beta)$ be given on the rectangle B, let continuous functions $\varphi(\alpha)$ and $\psi(\beta)$ be given on the intervals $[\alpha_0, \alpha_1]$ and $[\beta_0, \beta_1]$, respectively, and let $\varphi(\alpha_0) = \psi(\beta_0) = t_0$, $\varphi(\alpha_1) = \psi(\beta_1) = t_1$. Then there exist in B points (ξ_0, η_0) and (ξ_1, η_1) such that $\varphi(\xi_0) = \psi(\eta_0) = t_0$, $\varphi(\xi_1) = \psi(\eta_1) = t_1$, and, on the set of points (α, β) of the rectangle $B_1\{\xi_0 \leq \alpha \leq \xi_1;\ \eta_0 \leq \beta \leq \eta_1\}$ such that $\varphi(\alpha) = \psi(\beta)$, the function $f(\alpha, \beta)$ assumes all intermediate values between $f(\xi_0, \eta_0)$ and $f(\xi_1, \eta_1)$.

Moscow Aviation Institute
named after Sergo Ordzhonikidze

Received
11 III 1961

References Cited

1. S. N. Bernstein, *UMN*, **8**, 32 (1940).

2. Z. Opial, *Ann. Polon. Math.*, **7**, 2, 223 (1960).

3. G. Scorza-Dragoni, *Boll. Un. Mat. Ital.*, **14**, 225 (1935).

4. F. Surikova, *On a boundary-value problem for ordinary differential equations*, Candidate' s dissertation, Moscow State University, 1955.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.