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D. B. FUKS

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Abstract

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D. B. FUKS

ON HOMOTOPIC DUALITY

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B. Eckmann and P. Hilton (^{1}) observed that among homotopic concepts and facts there exists a certain duality. Thus, in the Eckmann–Hilton theory, cohomology groups are dual to homotopy groups, the direct product of spaces is dual to the bouquet of spaces, and the concept of a fibration in the sense of Serre is dual to the concept of a cofibration. If in some theorem of homotopy theory the concepts occurring in it are replaced by dual ones, then the theorem so obtained, as a rule, turns out to be true. For example, to the formula $\pi_j(X \times Y) = \pi_j(X) + \pi_j(Y)$ there is dual the formula $H^i(X \vee Y) = H^i(X) + H^i(Y)$. In the Eckmann–Hilton theory, not only the simplest concepts of homotopy topology are dualized, but also more complicated ones, for example the concept of the Lusternik–Schnirelmann category (see (^{2})).

However, the Eckmann–Hilton duality has essential defects. Up to now no exact definition of duality has been given, and in each separate case the definition of the dual object has to be given from intuitive considerations. If a concrete theorem of homotopy theory is given, then at best one succeeds in formulating its dual. But even if the assertion is dualized, the dual assertion does not follow from the original one, but requires an independent proof. There exist theorems which cannot be dualized, for example the well-known formula of Kunneth.

The purpose of the present note is to give, within certain limits, an exact definition of duality. This is achieved by means of a different approach to the theory. It is clear that in the Eckmann–Hilton duality one may speak not only of dual concepts, but also of dual functors. Examples of such dual functors are the direct product $X \times Y$ and the bouquet $X \vee Y$, the loop space ΩX and the suspension ΣX. We shall regard as basic not a duality between concepts, but a duality between functors. We begin with the fact that for every covariant functor S, acting from the category of topological spaces to the same category, we construct a dual functor DS. In an analogous way duality can also be defined for certain other types of functors. By a space we shall mean a Hausdorff space with a distinguished point; by a mapping of a space X into a space Y, a continuous mapping $f : X \to Y$ carrying the distinguished point of the space X into the distinguished point of the space Y.

One says that in the category of spaces a (covariant) functor S is given if to every space X there is assigned a space SX, and to every mapping $f : X \to Y$ there is assigned a mapping $Sf : SX \to SY$, with the following conditions satisfied:

1°. If $e : X \to X$ is the identity mapping of the space X, then $Se : SX \to SX$ is the identity mapping of the space SX.

2°. If $f : X \to Y$ and $g : Y \to Z$ are mappings, and $g \circ f : X \to Z$ is their composition, then $S(g \circ f) = Sg \circ Sf$.

3°. The correspondence $f \to Sf$ defines a continuous mapping of pro-

spaces of mappings of X into Y to the space of mappings of SX into SY (both in the compact-open topology).

4°. If X is a space consisting of a single point, then SX also consists of a single point.

Let us consider two important examples.

The functor Ω_A. Let A and X be spaces. Denote by X^A the space of mappings of A into X in the compact-open topology. If $f : X \to Y$ is a mapping, then $f^A : X^A \to Y^A$ is the mapping defined by the formula $f^A(\alpha) = f \circ \alpha$, where $\alpha \in X^A$. The functor Ω_A is defined by the equalities $\Omega_A X = X^A$ and $\Omega_A f = f^A$.

The functor Σ_A. Let A and X be spaces. The coordinate cross of the product $X \times A$ is the set of those points (x, a) $(x \in X,\ a \in A)$ for which at least one of the points x and a is distinguished in its space. In the direct product $X \times A$ identify all points of the coordinate cross with one another. Denote the resulting space by $X \# A$. Every mapping $f : X \to Y$ naturally induces a mapping $f \# A : X \# A \to Y \# A$. The functor Σ_A is defined by the equalities $\Sigma_A X = X \# A$ and $\Sigma_A f = f \# A$.

In what follows, when specifying a concrete functor S, we shall restrict ourselves to assigning to each space X the space SX, since in all cases that we shall consider a mapping $f : X \to Y$ will naturally define a mapping $Sf : SX \to SY$.

Let S and T be two functors. One says that a mapping α of the functor S into the functor T is given (notation: $\alpha : S \to T$) if, for every space X, a mapping $\alpha_X : SX \to TX$ is given, and if $f : X \to Y$ is a mapping, then

$$Tf \circ \alpha_X = \alpha_Y \circ Sf.$$

The set of all mappings of the functor S into the functor T will be denoted by $\{S \to T\}$, or by $\{SX \to TX\}_X$. For each X, the correspondence $\alpha \to \alpha_X$ defines a mapping

$$\varphi_X : \{S \to T\} \to TX^{SX}.$$

A point $\alpha \in \{S \to T\}$ is called a limit point of the set $M \subset \{S \to T\}$ if, for every X, the point $\varphi_X(\alpha)$ is a limit point of the set $\varphi_X(M) \subset TX^{SX}$ in the compact-open topology. Thus a Hausdorff topology is defined in $\{S \to T\}$.

Let S be a functor. Consider two functors: the functor S', which assigns to a space A the space $\{\Omega_A \to S\}$, and the functor S'', which assigns to a space A the space $\{S \to \Sigma_A\}$.

Proposition 1. *For every functor S, the functors S and S' coincide.*

Proof. Let $e : A \to A$ be the identity mapping, $\alpha \in \{\Omega_A \to S\}$. Then set $\lambda_A(\alpha) = \alpha_A(e)$. It is easy to verify that the mapping λ_A of the space $\{\Omega_A \to S\}$ into SA thus defined is continuous. Now construct a continuous mapping $\mu_A : SA \to \{\Omega_A \to S\}$. Let $x \in SA$. We must construct a mapping $\mu_A(x)$ of the functor Ω_A into the functor S. If X is a space and $f \in \Omega_A X$ is a mapping of A into X, then we set

$$(\mu_A(x))_X(f) = Sf(x).$$

We shall show that the mappings λ_A and μ_A are mutually inverse.

1°. The composition $\lambda_A\mu_A : S(A) \to S(A)$ is the identity mapping. Indeed,

$$\lambda_A(\mu_A(x)) = (\mu_A(x))_A(e) = Se(x) = x$$

by the definition of the functor.

2°. The composition $\mu_A\lambda_A : \{\Omega_A \to S\} \to \{\Omega_A \to S\}$ is the identity mapping. Indeed, let $\alpha \in \{\Omega_A \to S\}$, $f \in \Omega_A X$. We shall prove that

$$\alpha_X(f) = Sf(\alpha_A(e)).$$

Indeed, $\Omega_A f(e) = f$ (here $\Omega_A f : \Omega_A A \to \Omega_A X$) and

$$\alpha_X(f) = \alpha_X[\Omega_A f(e)] = Sf[\alpha_A(e)]$$

(the last equality holds by virtue of the definition of a mapping of functors). But

$$[\mu_A(\lambda_A(\alpha))]_X(f) = [\mu_A(\alpha_A(e))]_X(f) = Sf(\alpha_A(e)) = \alpha_X(f),$$

i.e. $\mu_A\lambda_A(\alpha) = \alpha$.

It is easy to check that the homeomorphism obtained commutes with the maps $S(f)$ and $S'(f)$. The proposition is proved.

The functor Ω_A in the Eckmann-Hilton theory is dual to the functor Σ_A. This suggests the following definition of a dual functor.

Definition. Let S be a functor. The functor $DS = S''$, assigning to a space A the space $\{S \to \Sigma_A\}$, is called the functor dual to it.

Examples.

1°. $D\Omega_A = \Sigma_A$.

2°. $D\Sigma_A = \Omega_A$.

3°. The suspension functor $X \to \Sigma X$ and the functor $X \to \Omega X$, where ΩX is the loop space of X, are dual to each other.

4°. The functor

$$X \to \underbrace{X \times \dots \times X}_{m}$$

and the functor

$$X \to \underbrace{X \vee \dots \vee X}_{m}$$

are dual to each other.

5°. Let X be a space. By $X * X$ we denote the join of X with itself, and by $J(X)$ the space of those paths in the bouquet $X \vee X$ which begin on one copy of the space X and end on the other. The functor $X \to X * X$ is dual to the functor $X \to J(X)$, and, conversely, the functor $X \to J(X)$ is dual to the functor $X \to X * X$.

Here 1° follows from Proposition 1; 2° and 5° are proved directly; 3° and 4° are special cases of 1° and 2° for $A = S^1$ (the circle) and for A consisting of $m+1$ isolated points.

Let S be an arbitrary functor. Then

$$DDS(Z) = \{\{SX \to \Sigma_{YX}\}_X \to \Sigma_{ZY}\}_Y.$$

We shall construct a map $\varkappa$ of the functor S into the functor DDS. Let Z be a space, $z \in SZ$. We define

$$\varkappa(z)_Y : \{S \to \Sigma_Y\} \to \Sigma_{ZY},$$

by putting, for $\alpha \in \{S \to \Sigma_Y\}$,

$$\varkappa(z)_Y(\alpha) = \alpha_Z(z).$$

Let S and T be arbitrary functors, and let $\alpha : S \to T$ be a map. Then, in a natural way, the dual map

$$D\alpha : DT \to DS$$

is constructed. Thus we obtain a map

$$J : \{S \to T\} \to \{DT \to DS\},$$

which turns out to be continuous. The equalities $S = DDS$ and $\{S \to T\} = \{DT \to DS\}$ for arbitrary functors are, generally speaking, false.

Theorem 1. For every functor S the equality

$$DS = DDDS$$

holds.

Proof. As noted above, for every functor T there exists a natural map

$$\varkappa : T \to DDT.$$

In particular, consider the maps

$$\varkappa_S : S \to DDS$$

and

$$\varkappa_{DS} : DS \to DDDS.$$

It turns out that the maps $\varkappa_{DS}$ and $D\varkappa_S : DDDS \to DS$ are mutually inverse.

Theorem 2. For any functors S and T the equality

$$\{DS \to DT\} = \{DDT \to DDS\}$$

holds.

Proof. Consider the map constructed above,

$$J,$$

substituting first DS and DT, and then DDT and DDS, in place of S and T. We obtain maps

$$\{DS \to DT\} \to \{DDT \to DDS\}$$

and

$$\{DDT \to DDS\} \to \{DDDS \to DDDT\}.$$

But by Theorem 1,

$$DDDS = DS$$

and

$$DDDT = DT,$$

therefore the second map may be regarded as a map

$$\{DDT \to DDS\}$$

into

$$\{DS \to DT\}.$$

It turns out that these maps are mutually inverse.

Definition. A functor S is called **reflexive** if

$$S = DDS.$$

It follows from Theorem 1 that a functor S is reflexive if and only if there exists a functor T such that

$$S = DT.$$

From Theorem 2 it follows that for reflexive functors S and T the equality

$$\{S \to T\} = \{DT \to DS\}$$

holds. Thus duality is “good”only on the set of reflexive functors. Unfortunately, some functors occurring very often in topology are not reflexive, for example the functor

$$X \to Z_{nX}$$

(Z_{nX} is the n-th cyclic power of the space X), the functor

$$X \to X\#X$$

and others. Indeed, one can···

show that both the functor $X \to X \# X$ and the functor $X \to Z_n X$ are dual to the zero functor, which assigns to each space X the space $*$, consisting of a single point.

We shall now define dualities for some other types of functors. Consider the category whose objects are continuous mappings $f : X_1 \to X_2$; a morphism φ of an object $f : X_1 \to X_2$ into an object $g : Y_1 \to Y_2$ is a pair (φ_1, φ_2), where $\varphi_1 : X_1 \to Y_1$, $\varphi_2 : X_2 \to Y_2$ are mappings such that $\varphi_2 f = g\varphi_1$ (notation $\varphi : f \to g$). By a functor in what follows we shall mean a covariant functor from this category to the category of spaces with an additional axiom of continuity, analogous to axiom 3^0 of a functor in the category of spaces.

Examples. Let $f : X_1 \to X_2$ be a mapping. Define the functor A_ν, $(\nu = 1, 2)$, by the equality $A_\nu(f) = X_\nu$. We shall also regard as functors the mapping cylinder Cf, the fiber Φf of the natural cofibration $i : X_1 \to Cf$. Serre $(^3)$ defined, for each mapping f, a fibration $p : K(f) \to X_2$ equivalent to it. The space $K(f)$ and the fiber Ff of this fibration will also be regarded as functors of f. Let $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$ be two mappings. Denote by f^g the space of all morphisms g into f in the natural topology, and by Ω the functor assigning to a mapping f the space f^g. Consider two spaces $X_1 \# Y_2$ and $X_2 \# Y_1$. Identify the points $(x, y') \in X_1 \# Y_2$ and $(x', y) \in X_2 \# Y_1$ if and only if $f(x) = x'$ and $y' = g(y)$. Denote the resulting space by $f \# g$. The functor Σ_g is defined by the equality $\Sigma_g f = f \# g$. In all the examples indicated, the mapping corresponding to a morphism is constructed in the natural way.

As in the case considered above, mappings of a functor into a functor, the space $\{U \to V\}$ of mappings of a functor U into a functor V, are defined. The equality $\{\Omega_f \to U\} = Uf$ turns out to be valid, suggesting the definition: the functor dual to a functor U is the functor DU assigning to a mapping f the space $\{U \to \Sigma_f\}$. The functors A_1 and A_2, Σ_f and Ω_f, turn out to be mutually dual; the functor C is dual to the functor K, and the functor Φ to the functor F. The last two examples, in a certain sense, express the duality of the notions of cofibration and fibration in the sense of Serre. For arbitrary functors U and V, mappings

$$\kappa' : U \to DDU$$

and

$$J' : \{U \to V\} \to \{DV \to DU\}$$

are constructed in a natural way, and theorems analogous to Theorems 1 and 2 hold.

In conclusion, we note that one can consider two further types of functors: respectively, from the categories of spaces and of mappings to the category of mappings. A functor of each of these types is naturally identified with a mapping of functors of the previously considered types, and the duality here is established with the aid of the mappings J and J'.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.

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