ON TRANSPORTATION OVER ROAD NETWORKS WITH LOSSES

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Abstract

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S. M. Shvartin

ON TRANSPORTATION OVER ROAD NETWORKS WITH LOSSES

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Let V be a directed connected network ([1]). Let $G = \{g\}$ be the set of possible states of a vertex of the network V. We shall call a mapping $\mu(x)$ of the set of vertices of the network V into G a state of the network V. We shall call a network V for which a mapping $\mu(x)$ is specified a loaded network. Let $\Xi = \{\mu(x)\}$ be the set of all possible states of the network V. Let $\mu_0(x) \in \Xi$ and $\Xi_1 \subset \Xi$ be given. We shall call the symbol $W = (V, \Xi, \mu_0(x), \Xi_1)$ a definite loaded network, $\mu_0(x)$ the initial state of the network, and Ξ_1 the set of terminal states of the network. Let a class

$$\mathcal{F} = \{F(g, g') = (\bar{g}, \bar{g}') : G^2 \to G^2\}$$

of functions be given, called transfer functions. A function $F \in \mathcal{F}$ and an edge $(x_i, x_j)_k \in W$ generate an elementary transfer operator A_k on the state of the network W

$$A_k(\mu(x)) = \mu_1(x) = \begin{cases} \bar{g}, & \text{if } x = x_i, \\ \bar{g}', & \text{if } x = x_j, \\ \mu(x), & \text{if } x \neq x_i \text{ and } x \neq x_j. \end{cases}$$

Let a class

$$\Psi = \{\psi(g) = \bar{g} : G \to G\}$$

of functions be given, called expressed transfer functions. A function $\psi \in \Psi$ and a vertex $x_i \in W$ generate an elementary expressed transfer operator

$$B_i(\mu(x)) = \mu_1(x) = \begin{cases} \bar{g}, & \text{if } x = x_i, \\ \mu(x), & \text{if } x \neq x_i. \end{cases}$$

Let

$$H_k = \{[(x, x')_k, F]\}$$

be the set of ordered pairs composed of one edge of the network and transfer functions from $\mathcal{F}$. Let

$$H = \bigcup_{k=1}^{\alpha_0} H_k.$$

Let

$$N_i = \{[x_i, \psi]\}$$

be the set of ordered pairs composed of one vertex of the network and a function $\psi \in \Psi$. Let

$$N = \bigcup_{i=1}^{\alpha_1} N_i$$

(α_0 and α_1 are the numbers of edges and vertices of the network W, respectively). We shall call the symbol $S = (W; H, N)$ a fully specified loaded network (hereafter called simply a network). It is required to construct a sequence of operators, generated by ordered pairs from H and N, that takes the network from the state $\mu_0(x)$ into a state $\mu(x) \in \Xi_1$. If the solution is not unique, then additional requirements of various kinds may be imposed on the sequence of operators. We shall call a network a network transmitting mass if all $g \in G$ are nonnegative real numbers. We shall call a network a network transmitting information if all $g \in G$ are ordered sequences of code symbols. In what follows –

networks transmitting mass are considered. We shall call the quantity $d_F = (\bar{g}+\bar{g}')-(g+g')(d_\psi = \bar{g}-g)$ the **mass defect** of the transmission function $F(\psi)$. We shall call a network a **network with losses** if $d_F < 0$ and $d_\psi = 0$ for all F and ψ ordered pairs from H and N, respectively. In what follows only networks with losses are considered.

We shall call a network **special** if

1) $\Xi = \{\mu(x) : \mu(x_i) \geq 0\}$.
2) $\Xi_1 = \{\mu(x) : \mu(x_i) \geq a_i\}$ ($a_i \geq 0$ is a prescribed real number) ($i = 1, 2, \dots, \alpha_1$).
3) For every $m \geq 0$, in each $H_k \in H$ there is an ordered pair $[(x, x')_k,\ F(g, g') = (\bar{g}, \bar{g}')]$ such that $g - \bar{g} = m$.
4) For all F ordered pairs from $\bar{H}_k$, $d_F = f_k(g - \bar{g})$; $g \geq \bar{g},\ g' \leq \bar{g}'$.

We shall call the function

$$\varphi_k(g - \bar{g}) = (g - \bar{g}) + f_k(g - \bar{g})$$

the **transmission function** of the edge $(x, x')_k$. In what follows special networks S are considered. Let
$A = A_{k_1}, A_{k_2}, \dots, A_{k_n}$ be a sequence of elementary operators on the state of the network S. We shall call the functional

$$P(A) = -\sum_{i=1}^{n} dA_{k_i}$$

the **loss functional** (dA_{k_i} is the mass defect of the operator $A_{k_i} \in A$). We shall call the sequence A **optimal** if it transfers the network S from the state

$\mu_0(x)$ to the state $\mu(x) \in \Xi_1$ and makes the functional $P(A)$ minimal. We shall call the loaded network $\bar{S}$ a **working subnet** of the network S if it is obtained from the network S by deleting edges that do not generate elementary operators of the sequence A, optimal for the network S, and its state is the state of the network S obtained as a result of applying the sequence A to the network S. We shall call a network a **network with linear losses** if
$\varphi_k(g-\bar{g}) = p_k(g-\bar{g})$ $(0 \leq p_k < 1;\ k = 1, 2, \dots, \alpha_0)$, and denote it by S_L. Let A be an optimal sequence for the network S_L; $\bar{S}_L$ its working subnet, and $\mu(x)$ the state of the working subnet $\bar{S}_L$.

Lemma 1. For all vertices x_i of the network S_L such that $\mu_0(x_i) \geq a_i$, $\mu(x_i) \geq a_i$, and for all x_j such that $\mu_0(x_j) \leq a_j$, $\mu(x_j) = a_j$.

Lemma 2. Let $(x_i, x_j)_{k_s}$ $(s = 1, 2, \dots, n)$ be edges of the network S_L; then the network $\bar{S}_L$ contains no more than one of these edges.

Lemma 3. The network $\bar{S}_L$ contains no loops.

Theorem 1. Let $(x_i, x_j)_k,\ (x_j, x_i)_s$ be edges of the network S_L. The network $\bar{S}_L$ contains no more than one of these edges, i.e. counter transmissions of masses are absent.

Theorem 2. The network $\bar{S}_L$ contains no circuits and, consequently, is a tree*.

Let Z_i^0 be the star of the vertex x_i of the network S_L (2). Attach to Z_i^0 all vertices of the network S_L incident with its edges. We shall call these vertices the **boundary vertices of the star**. We shall call Z_i^0, together with the boundary, a **closed star** and denote it by Z_i. We shall say that the vertex x_i of the network S_L belongs to class K_1, K_2, K_3 if $\mu_0(x_i) > a_i$, $\mu_0(x_i) < a_i$, $\mu_0(x_i) = a_i$, respectively. We shall say that the star Z_i of the network S_L belongs to class K_{sp}, K_{spq} $(p < q)$, K_{s123}, if $x_i \in K_s$ and its boundary vertices belong only to classes K_p; K_p and K_q; K_1, K_2, and K_3, respectively $(s, p = 1, 2, 3;\ q = 2, 3)$. Each star Z_i of the network S_L belongs to one of the 21 possible classes of stars.

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* An analogous result for the case of transport without losses was obtained by M. L. Tsetlin (3).

Theorem 3. Every star Z_i of the network $\overline{S}_L$ belongs to one of the 4 classes $K_{13}, K_{31}, K_{33}, K_{313}$; in a star $Z_i \in K_{13}$ each boundary vertex is the end of an edge of the star; in a star $Z_i \in K_{313}$ there is only one boundary vertex of class K_1. This vertex is always the beginning of an edge of the star Z_i.

Let the network S_L be a tree. Let $x_i \in K_2$ and be an end vertex of the network S_L, and let $(x_i, x_j)_s$ be an edge of the network S_L. Define the operator $\mathfrak{A}^-$ on the state of the network:

$$\mathfrak{A}^{-}(\mu(x)) = \mu_1(x) = \begin{cases} a_i, & \text{if } x = x_i; \\ \mu(x_j) - \dfrac{a_i - \mu(x_i)}{p_s}, & \text{if } x = x_j; \\ \mu(x), & \text{if } x \neq x_i \text{ and } x \neq x_j. \end{cases}$$

Let $x_i \in K_1$ and be an end vertex of the network S_L. Let $(x_i, x_j)_k$ be an edge of the network. Define the operator $\mathfrak{A}^{+}$ on the state of the network:

$$\mathfrak{A}^{+}(\mu(x)) = \mu_1(x) = \begin{cases} a_i, & \text{if } x = x_i; \\ \mu(x_j) + [\mu(x_i) - a_i]p_k, & \text{if } x = x_j; \\ \mu(x), & \text{if } x \neq x_i \text{ and } x \neq x_j. \end{cases}$$

Let $x_i \in K_3$ be an end vertex of the network S_L, and let $(x_i, x_j)_k$ and $(x_j, x_i)_s$ be edges of the network S_L. Denote by $\mathfrak{A}$ the operator that excludes these edges from the network S_L. Let Z_i be the closed star of the vertex x_i of the network S_L. We shall call **folding the network S_L into the star Z_i** the operation consisting in applying the operators $\mathfrak{A}, \mathfrak{A}^{+}$, and $\mathfrak{A}^{-}$ to the network S_L as long as this is possible under the prohibition on considering boundary vertices of the star as end vertices. Let S'_L be the closed star Z_i of the vertex x_i. We shall call the star **balanced** if the operators $\mathfrak{A}^{+}$ and $\mathfrak{A}^{-}$ carry this star from its state into one of the final states. Otherwise the star Z_i is called **unbalanced**. Let Z_i be the closed star of the vertex x_i of the network S_L. Then the following hold:

Theorem 4. Let the network S_L be folded into the star Z_i, and let the star Z_i be unbalanced. Then every star Z_k of the network S_L is unbalanced after folding the network S_L into it.

Theorem 5. For the existence of an optimal sequence of operators on the network S_L, it is necessary and sufficient that it contain a closed star which becomes balanced after folding the network S_L into it.

Let S_p be a special network of arbitrary nature with nondecreasing concave transfer functions of the edges (i.e.

$$\varphi_k\left(\frac{x_1 + x_2}{2}\right) \geq \frac{1}{2}\left[\varphi_k(x_1) + \varphi_k(x_2)\right] \quad \text{and} \quad 0 \leq \varphi_k(x) \leq x.$$

Theorem 6. For the network S_p, there is no optimal sequence of operators.

Let S'_L be a network with linear losses obtained from the network S_p by replacing the transfer functions $\varphi_k(g - \overline{g})$ of the edges of the network S_p by the linear transfer functions

$$\left.\frac{d\varphi_k}{d(g - \overline{g})}\right|_{(0+0)} \cdot (g - \overline{g}).$$

Let A'_L be an optimal sequence of operators on the network S'_L, and let A_p be an arbitrary sequence of operators on the state of the network S_p that carries it from the initial state to one of the final states. Then the following holds:

Theorem 7. $P(A_L) \leqslant P(A_p)$, and for every $\varepsilon > 0$ there exists a sequence of operators A_p on the network S_p that transfers it from the initial state to the state $\mu(x) \in \Xi_1$, such that $P(A_p) - P(A_L) < \varepsilon$.

Let S_q be a special network of arbitrary nature with nondecreasing convex transfer functions of the edges (i.e.,

$$\varphi_k\left(\frac{x_1 + x_2}{2}\right) \leqslant \frac{1}{2}\left[\varphi_k(x_1) + \varphi_k(x_2)\right]$$

and $0 \leqslant \varphi_k(x) \leqslant x$.

For such networks the optimal sequence may contain counter-transfers, and the working subnet may contain contours. The property determined by Lemma 1 does not hold. Let $\ldots, t_{-n}, \ldots, t_{-1}, t_0, t_1, \ldots, t_n, \ldots$ be a discrete time scale. Let S be an arbitrary network. We shall call the symbol (S, t_0) a network with initial state $\mu_0(x)$ at time t_0. We shall assume that an elementary operator $A_k^t(\mu(x)) = \mu_1(x)$, acting on the network S at time t, transfers it to the state $\mu_1(x)$ at time $t + 1$.

Here the following problems are considered:

1) Given a time instant $t_1 > t_0$. It is required to construct a sequence of operators A_k^t that transfers the network (S, t_0) from the state $\mu_0(x)$ at time t_0 to a state $\mu(x) \in \Xi_1$ at a time $t \leqslant t_1$ and minimizes the loss functional.

2) It is required to construct a sequence of operators that transfers the network (S, t_0) to a state $\mu(x) \in \Xi_1$ in the least possible time.

Under these conditions the optimal sequences of operators may contain counter-transfers, and the working subnets may contain contours. These properties are determined not by the nature of the transfer functions of the edges, but exclusively by the topology of the network and by the boundedness of time.

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CITED LITERATURE

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