THEORY OF ELASTICITY

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Abstract

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SOME NONLINEAR BOUNDARY-VALUE PROBLEMS IN THE THEORY OF A MAXWELL BODY

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Some physical problems concerning the motion of an elastic body in which stresses relax lead to the determination of the displacement vector of the body $\mathbf{u}(x_1, x_2, x_3, t)$, with components u_1, u_2, u_3, and of the stress tensor $\sigma_{x_i x_j}(x_1, x_2, x_3, t)$, $i, j = 1, 2, 3$, which satisfy the following system of equations:

$$\vec{\sigma} = N_1 \vec{\varepsilon} - N_2 \int_0^t \exp\left[-\int_\tau^t \frac{ds}{T}\right] \frac{2\mu}{3T} \vec{\varepsilon}\, d\tau,$$

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \rho \mathbf{F} = \vec{\Phi}_t(\mathbf{u}), \tag{1}$$

where $\vec{\varepsilon}$ is the column with components $\partial u_1/\partial x_1,\ \partial u_2/\partial x_2,\ \partial u_3/\partial x_3,\ \partial u_1/\partial x_2 + \partial u_2/\partial x_1,\ \partial u_1/\partial x_3 + \partial u_3/\partial x_1,\ \partial u_2/\partial x_3 + \partial u_3/\partial x_2$; $\vec{\sigma}$ is the column with components $\sigma_{x_1x_1}, \sigma_{x_2x_2}, \sigma_{x_3x_3}, \sigma_{x_1x_2}, \sigma_{x_1x_3}, \sigma_{x_2x_3}$; λ, μ are positive constants (Lamé constants); $\rho(x_1, x_2, x_3)$ is a positive function (density); $\mathbf{F}(x_1, x_2, x_3, t)$ is the vector with components F_1, F_2, F_3 (the body-force vector); $T(x_1, x_2, x_3, t, \vec{\sigma})$ is a positive function (the relaxation time); and N_1 and N_2 are the following matrices of order 6:

$$N_1 = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3/2 \end{pmatrix}. \tag{2}$$

The vector $\vec{\Phi}_t(\mathbf{u})$ has the following three components $\Phi_{it}(\mathbf{u})$:

$$\Phi_{it}(\mathbf{u}) = \frac{\partial}{\partial x_i} \int_0^t \exp\left[-\int_\tau^t \frac{ds}{T}\right] \frac{2\mu}{3T} \left(2\frac{\partial u_i}{\partial x_i} - \frac{\partial u_j}{\partial x_j} - \frac{\partial u_k}{\partial x_k}\right) d\tau + \tag{3}$$

$$+\frac{\partial}{\partial x_j}\int_0^t \exp\left[-\int_\tau^t \frac{ds}{T}\right]\frac{\mu}{T}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)d\tau+\frac{\partial}{\partial x_k}\int_0^t \exp\left[-\int_\tau^t \frac{ds}{T}\right]\frac{\mu}{T}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)d\tau,$$

where the indices i, j, k take the values $1, 2, 3$ and are obtained from $1, 2, 3$ by cyclic permutation.

In what follows we shall assume that the point x_1, x_2, x_3 belongs to a bounded domain D of three-dimensional space and that the boundary Γ of the domain D is a surface whose curvature is continuous. In addition, we shall suppose that $\Gamma = \Gamma_1 + \Gamma_2$, and that the boundary separating Γ_1 and Γ_2 is a smooth curve.

We shall consider the following three boundary-value problems.

1. The first boundary-value problem. Find the vector $\mathbf{u}(x_1, x_2, x_3, t)$ and the stress tensor $\sigma_{x_i x_j}(x_1, x_2, x_3, t)$ satisfying equations (1) and the boundary condition

$$\mathbf{u}(s) = \vec{\varphi}(s,t), \qquad s \in \Gamma,$$

where $\vec{\varphi}(s,t)$ is a given vector.

2. Find the vector $\mathbf{u}(x_1, x_2, x_3, t)$ and the stress tensor $\sigma_{x_i x_j}(x_1, x_2, x_3, t)$, satisfying equations (1) and the boundary conditions

$$\sigma_{x_i x_1}\cos nx_1 + \sigma_{x_i x_2}\cos nx_2 + \sigma_{x_i x_3}\cos nx_3\Big|_\Gamma = X_i(s,t), \qquad s \in \Gamma, \quad i = 1, 2, 3,$$

where $\cos nx_1,\ \cos nx_2,\ \cos nx_3$ are the direction cosines of the outward normal to the boundary Γ; $X_i(s,t)$ is a given vector (the vector of external forces).

3. Find the vector $\mathbf{u}(x_1, x_2, x_3, t)$ and the tensor $\sigma_{x_i x_j}(x_1, x_2, x_3, t)$, satisfying equations (1) and, on Γ_1, boundary conditions 1, while on Γ_2, boundary conditions 2.

We shall define the generalized solution of the boundary-value problems posed. To this end we introduce the Hilbert space $\mathfrak{R}$ and the normed space $\mathfrak{R}(t_1, t_2)$ as follows. $\mathfrak{R}$ is the orthogonal sum of two Hilbert spaces $L_2(D)$ and $\mathfrak{H}(D)$, where $L_2(D)$ is the Hilbert space of vectors with 6 components and with modulus whose square is integrable over the domain D, while $\mathfrak{H}(D)$ is one of the three Hilbert spaces $H_I(D), H_{\mathrm{II}}(D), H_{\mathrm{III}}(D)$, introduced in the works ([1,2]). Recall that the scalar product in these spaces is the expression

$$W(\mathbf{u},\mathbf{v}) = \iiint_D \left[\lambda\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)\right.$$
$$+ 2\mu\frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + 2\mu\frac{\partial u_2}{\partial x_2}\frac{\partial v_2}{\partial x_2} + 2\mu\frac{\partial u_3}{\partial x_3}\frac{\partial v_3}{\partial x_3} + \mu\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)\left(\frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1}\right)$$
$$\left. + \mu\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)\left(\frac{\partial v_1}{\partial x_3}+\frac{\partial v_3}{\partial x_1}\right) + \mu\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)\left(\frac{\partial v_2}{\partial x_3}+\frac{\partial v_3}{\partial x_2}\right)\right] dx_1\,dx_2\,dx_3. \tag{4}$$

Here the space $H_I(D)$ is obtained by closure, in the scalar product (4), of the linear set of continuously differentiable vectors $\mathbf{u}$ vanishing on Γ; $H_{\mathrm{III}}(D)$ is the closure, in the scalar product (4), of the linear set of continuously differentiable vectors vanishing on Γ_1, while $H_{\mathrm{II}}(D)$ is the closure in (4) of the linear set of continuously differentiable vectors satisfying the conditions

$$\iiint_D \mathbf{u}\,dx_1\,dx_2\,dx_3 = 0, \qquad \iiint_D (\mathbf{R}\times\mathbf{u})\,dx_1\,dx_2\,dx_3 = 0,$$

where $\mathbf{R}$ is the radius vector of the point x_1, x_2, x_3. The scalar product in $\mathfrak{R}$ will be denoted by (,), and the norm by $\|\ \|$. The projection operators from $\mathfrak{R}$ onto $\mathfrak{H}$ and $L_2(D)$ will be denoted by $P_{\mathfrak{H}}$ and P_{L_2}.

By $\mathfrak{R}(t_1, t_2)$ we shall denote the linear normed space of vector-functions with values in $\mathfrak{R}$, defined on the interval $t_1 \le t \le t_2$, with norm

$$\left\|\vec{\psi}\right\|_{(t_1,t_2)} = \sup_{t_1\le t\le t_2}\left\|\vec{\psi}(t)\right\|.$$

$\mathfrak{R}(t_1, t_2)$ is a complete space, convergence in which means uniform convergence on the interval $t_1 \le t \le t_2$ of vector-functions in the norm of $\mathfrak{R}$.

Multiplying the first equation (1) scalarly by an arbitrary vector $\vec{\varphi} \in L_2(D)$, the second by $\mathbf{v} \in \mathfrak{H}$, integrating on the left over the domain D, integrating the latter equation by parts, and denoting by $\mathbf{u}_2$ the solution of the equation

$$\mu\Delta\mathbf{u}_2 + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}_2 + \varrho\mathbf{F} = 0 \tag{5}$$

with one of the boundary conditions

1′. $\mathbf{u}_2|_\Gamma = \vec{\varphi}(s,t)$,
2′. $\sigma_{2x_ix_1}\cos nx_1 + \sigma_{2x_ix_2}\cos nx_2 + \sigma_{2x_ix_3}\cos nx_3|_\Gamma = X_i(s,t), \quad i = 1,2,3,$

where

$$\sigma_{2x_ix_j} = \mu\left(\frac{\partial u_{2i}}{\partial x_j} + \frac{\partial u_{2j}}{\partial x_i}\right) + \lambda\delta_{ij}\,\mathrm{div}\,\mathbf{u}_2.$$

$3'$. $\Gamma = \Gamma_1 + \Gamma_2$, on Γ_1 condition $1'$ holds, and on Γ_2 condition $2'$ holds.

Denoting by $\vec{\psi}$ the pair $\{\mathbf{u}_1, \vec{\sigma}\}$, where $\mathbf{u}_1 = \mathbf{u} - \mathbf{u}_2$, we can give the following

Definition. A solution of equations (1) with one of the boundary conditions 1–3 is a pair $\{\mathbf{u}, \vec{\sigma}\}$ such that the pair $\vec{\psi}_1 = \{\mathbf{u}_1, \vec{\sigma}\}$ satisfies, for every $\vec{\psi} \in \mathfrak{R}$, the functional equation

$$\begin{aligned}(\vec{\psi}_1, \vec{\psi}) = &\iiint_D [N_1 P_{\mathfrak{H}}\vec{\psi}_1,\ P_{L_2}\vec{\psi}]\,dx_1dx_2dx_3\\
&- \iiint_D \left[N_2\int_0^t \exp\left[-\int_\tau^t \frac{ds}{T}\right]\frac{2\mu}{3T}P_{\mathfrak{H}}\vec{\psi}_1\,d\tau,\ P_{L_2}\vec{\psi}\right]dx_1dx_2dx_3\\
&+ \iiint_D \left[N_2\int_0^t \exp\left[-\int_\tau^t \frac{ds}{T}\right]\frac{2\mu}{3T}P_{\mathfrak{H}}\vec{\psi}_1\,d\tau,\ P_{\mathfrak{H}}\vec{\psi}\right]dx_1dx_2dx_3\\
&- \iiint_D \left[N_2\int_0^t \exp\left[-\int_\tau^t \frac{ds}{T}\right]\frac{2\mu}{3T}\vec{\varepsilon}_2\,d\tau,\ P_{L_2}\vec{\psi}\right]dx_1dx_2dx_3\\
&+ \iiint_D \left[N_2\int_0^t \exp\left[-\int_\tau^t \frac{ds}{T}\right]\frac{2\mu}{3T}\vec{\varepsilon}_2\,d\tau,\ P_{\mathfrak{H}}\vec{\psi}\right]dx_1dx_2dx_3,\end{aligned} \tag{6}$$

$\vec{\varepsilon} = \vec{\varepsilon}_1 + \vec{\varepsilon}_2$; $\vec{\varepsilon}_1$ and $\vec{\varepsilon}_2$ are the columns corresponding to the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

Consider an arbitrary t_0 and divide the interval $0 \le t \le t_0$ into n equal parts $\tau_i \le t \le \tau_{i+1}$, $i = 0, 1, 2, \ldots, n$, $\tau_0 = 0$, $\tau_{n+1} = t_0$, and set

$$T(x_1, x_2, x_3, \vec{\sigma}(t)) = T(x_1, x_2, x_3, \vec{\sigma}(\tau_i)), \quad \tau_i \le \tau \le \tau_{i+1}.$$

Then the equations with boundary conditions 1–3 become linear on each of the intervals $\tau_i \le \tau \le \tau_{i+1}$, and from estimates of the right-hand sides of (6) it follows that the boundary-value problems (1) have a unique solution belonging to $\mathfrak{R}(0, t_0)$. The solution $\vec{\psi}_{1n} \in \mathfrak{R}(0, t_0)$ found in this way will be called the Euler polygonal line of the boundary-value problem (1), 1-3.

Lemma 1. *The family of Euler polygonal lines is uniformly bounded and equicontinuous in the norm in* $\mathfrak{R}$ *on the interval* $0 \le t \le t_0$*, if the conditions*

$$\sup_{0\le t\le t_0} \mathrm{W}(\mathbf{u}_2,\mathbf{u}_2)<\infty, \qquad \inf_{\substack{x\in D\\ 0\le t\le t_0\\ -\infty<|\vec\sigma|<\infty}} T(x_1,x_2,x_3,t,\vec\sigma)>0 \tag{7}$$

are satisfied.

The proof of the lemma is based on applying the Cauchy-Bunyakovsky inequality and conditions (7) to the right-hand side of (6).

Theorem 1. *Suppose that conditions (7) are satisfied. Suppose, in addition, that*

$$\sup_{0\le t\le t_0}\left|\frac{\partial}{\partial t}W(\mathbf{u}_2,\mathbf{u}_2)\right|<\infty, \qquad \sup_{\substack{x\in D\\ 0\le t\le t_0\\ -\infty<|\vec\sigma|<\infty}}\left|\frac{\partial T}{\partial t}\right|<\infty, \qquad \sup_{\substack{x\in D\\ 0\le t\le t_0\\ -\infty<|\vec\sigma|<\infty}}\left|\frac{\partial T}{\partial \vec\sigma}\right|<\infty. \tag{8}$$

Then equation (6) has a unique solution belonging to $\mathfrak{R}(0,t_0)$.

Idea of the proof. Choose on the interval $0\le t\le t_0$ an everywhere dense countable sequence t_i; for each t_i choose a weakly convergent sequence of Euler polygonal lines $\vec\psi_{1n_1n_2\ldots n_{i-1}n_i}$ from the already chosen subsequence $\vec\psi_{1n_1n_2\ldots n_{i-1}}$, converging at the points $t_1,t_2,\ldots,t_{i-1}$. This can always be done by virtue of the uniform boundedness of the family $\vec\psi_{1n}$. From the sequences $\vec\psi_{1n_1n_2\ldots n_s}$ choose a diagonal subsequence $\vec\psi_{1n}$, converging weakly, by virtue of the uniform continuity of the family $\vec\psi_{1n}$, to some limit $\vec\psi_1$. Estimates of the right-hand side of (6) imply that this limit is a solution of (6). If, however, there exist two solutions $\mathbf{u}_1$ and $\mathbf{v}_1$, then from the estimates of the right-hand side of (6) the relation follows

$$\left|(\vec\psi_1-\vec\varphi_1,\vec\psi)\right|<ct_0\left|(\vec\psi_1-\vec\varphi_1,\vec\psi)\right|, \qquad \psi\in\mathfrak{R},$$

where the constant c does not depend on t. Hence the uniqueness of the solution follows.

To find an approximate solution of (6), one may use a Galerkin-type method. Let $\mathbf{g}_i$ be a basis in $\mathfrak{R}$. The approximate solution

$$\vec\psi^{(n)}(t)=\sum_{i=1}^{n}c_i^{(n)}(t)\mathbf{g}_i$$

is found from the system of equations

$$c_k^{(n)}(t)=\Phi_t\left(\sum_{i=1}^{n}c_i^{(n)}(t)\mathbf{g}_i,\mathbf{g}_k\right), \qquad k=1,2,\ldots,n, \tag{9}$$

where $\Phi_t(\vec{\psi}, \vec{\psi})$ denotes the right-hand side of (6).

Theorem 2. *Suppose that conditions (7), (8) are satisfied. Then system (9) has a solution, and moreover a unique one. In addition, for any* $\vec{\psi} \in \mathfrak{R}$ *and uniformly on the whole interval* $0 \le t \le t_0$,

$$(\vec{\psi}_1 - \vec{\psi}_1^{(n)}, \vec{\psi}) \to 0, \qquad n \to \infty,$$

where $\vec{\psi}_1$ *is the solution of equation (6).*

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CITED LITERATURE

1. S. G. Mikhlin, *The Problem of the Minimum of a Quadratic Functional*, 1952.

2. B. M. Naimark, *Transactions of the Institute of Physics of the Earth*, No. 11 (178) (1959).

Note: Figure translations are in progress. See original paper for figures.