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Abstract

Full Text

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ON A LOCAL AND NONLOCAL EXISTENCE THEOREM FOR NONLINEAR PARABOLIC EQUATIONS OF SECOND ORDER

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1. Let Ω be a bounded domain of m-dimensional space with boundary S. We consider the question of the existence of a solution of the equation*

$$v'_t - a_{ik}(t, x, v, v'_{x_1}, \dots, v'_{x_m})v''_{x_i x_k} = f(t, x, v, v'_{x_1}, \dots, v'_{x_m}), \tag{1}$$

satisfying the initial condition

$$v(0, x) = v_0(x) \tag{2}$$

and, on the boundary S, one of the conditions

$$v(t, x) = 0 \tag{3}$$

or

$$a_{ik}(t, x, v, v'_{x_1}, \dots, v'_{x_m})v'_{x_k} \cos(n, x_i) + \sigma(t, x, v, v'_{x_1}, \dots, v'_{x_m})v = 0.$$

Here n is the vector normal to the surface S.

The boundary S, as well as the functions entering into equation (1) and the boundary conditions (3), are assumed to be sufficiently smooth. The form $a_{ik}\gamma_i\gamma_k$ is assumed to be positive definite, and the function σ nonnegative.

In a number of recent works, various particular types of such a problem have been investigated (see, for example, ([1]) and the bibliography given there).

The mixed problem (1), (2), (3) for a partial differential equation can be reduced ([2,3]) to the Cauchy problem for an ordinary differential equation

$$v' + A(t, v)v = f(t, v), \qquad v(0) = v_0, \tag{4}$$

in some functional space, for example the space $L_p(\Omega)$.

For each function v from a certain set, $A(t, v)$ is a linear operator acting in $L_p(\Omega)$, defined by the elliptic differential expression**

$$-a_{ik} z''_{x_i x_k} \tag{5}$$

and by one of the boundary conditions

$$z = 0 \quad \text{or} \quad a_{ik} z'_{x_k} \cos(n, x_i) + \sigma z = 0. \tag{6}$$

2. It can be shown that the operator $A(t, v) + \lambda I$ has a bounded inverse for every λ with $\operatorname{Re}\lambda \geq \lambda_0$, and that the estimate

$$\left\| [A(t, v) + \lambda I]^{-1} \right\|_{L_p} \leq C_p [|\lambda| + 1]^{-1} \tag{7}$$

holds.

* Here and below summation over identical indices from 1 to m is understood.

** The functions a_{ik}, σ depend on t, x, v, v'_{x_i}.

Operators possessing such properties were studied in ([4], [5]). In ([5], [6]) it is shown that, for such operators, one can construct fractional powers possessing properties analogous to the properties of positive-definite self-adjoint operators in Hilbert space.

If the first of the boundary conditions (6) is satisfied, then the domain of definition $D[A(t, v)]$ of the operator $A(t, v)$ does not depend on t and v. In the case where the second of the boundary conditions (6) is satisfied, one can show that $D[A^{1/2+\varepsilon}(t, v)] \subset D[A^{1/2-\varepsilon}(\tau, w)]$ for every $\varepsilon > 0$.

The theory developed in ([3]) for equations in Hilbert space with a positive-definite self-adjoint operator carries over to differential equations (4) with such operators.

Denote by $U_v(t, 0)v_0$ the solution of the linear homogeneous problem

$$z' + A(t, v)z = 0, \qquad z(0) = v_0. \tag{8}$$

With the aid of the operator $U_v(t, v)$, the nonlinear problem (4) is reduced ([2], [3]) to the nonlinear problem

$$v(t) = U_v(t, 0)v_0 + \int_0^t U_v(t, s) f(s, v)\, ds. \tag{9}$$

It can be shown that, for sufficiently small $t_0 > 0$, the operator

$$U_v(t,0)v_0 + \int_0^t U_v(t,s) f(s,v)\,ds$$

maps the set of functions $v(t,x)$ satisfying the initial condition (2) and, together with their first derivatives with respect to the spatial variables, satisfying a certain Hölder condition in all variables for $0 \le t \le t_0$, $x \in \bar{\Omega}$, into its compact part and that, moreover, this operator is continuous. From Schauder' s principle there then follows the local theorem on existence of a solution of problem (4) and the local theorem on existence of a classical solution of problem (1), (2), (3). We note that the existence of a solution of the mixed problem (1), (2), (3) can be established without any restrictions on the growth of the nonlinearities.

3. An essential role in the proof of these facts is played by the question of in which spaces among $L_p(\Omega)$ the operators

$$A^{-\rho}(t,v), \qquad \frac{\partial}{\partial x_i} A^{-\rho}(t,v)$$

act boundedly (or completely continuously).

We note that the question of such properties of fractional powers of operators was first posed and studied in ([7]). In the case of the first of the boundary conditions (6), the answer to this question follows from general theorems ([8], [9]).

In the case of the second of the boundary conditions (6), we first construct the semigroup $e^{-\tau A(t)}$ $(\tau \ge 0)$ with the aid of the Green' s function for the parabolic equation. The existence of the Green' s function and the necessary properties are established by classical methods with the aid of integral equations (see, for example, ([10])). Finally, the negative fractional powers $A^{-\rho}(t,v)$ are obtained with the aid of the formula for the gamma function (see ([11])).

4. It turns out that the quantity t_0 ultimately depends on the magnitude of $\|A^{\rho}(0,v_0)v_0\|_{L_p}$ for some $\rho \in [0,1)$ and sufficiently large p. Therefore, if one succeeds in establishing the a priori estimate

$$\sup_{0 \le t \le T} \|A^{\rho}(t,v)v\|_{L_p} \le C_{\rho}(T), \tag{10}$$

then a nonlocal existence theorem is valid*.

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* Such an approach to the proof of nonlocal existence theorems was developed earlier by M. A. Krasnosel' skii, S. G. Krein, and P. E. Sobolevskii (see ([13])).

We shall establish the indicated a priori estimate for classical solutions of the simpler equation

$$v'_t - a_{ik}(t,x)v''_{x_i x_k} + a_i(t,x,v)v'_{x_i} + a(t,x,v) = 0 \tag{11}$$

and of the simpler boundary conditions

$$v(t,x) = 0 \quad \text{or} \quad a_{ik}(t,x)v'_{x_k}\cos(n,x_i) + \sigma(t,x,v)v = 0. \tag{12}$$

Lemma. For any $\varepsilon > 0$ the inequality* holds

$$\|v'_{x_i}\|_{L_p} \le \varepsilon\|A(t)v\|_{L_p} + c_p(\varepsilon,T)\|v\|_{L_p}. \tag{13}$$

The proof is carried out by the methods indicated in item 3.

5. We shall assume that the one-sided estimate

$$a(t,x,v)v \ge -M(T)v^2 - N(T) \qquad (0 \le t \le T,\ x \in \overline{\Omega}) \tag{14}$$

is satisfied. Then it follows easily from the maximum principle that

$$\max_{t\in[0,T],\, x\in\overline{\Omega}} |v(t,x)| \le A(T)\max_{x\in\overline{\Omega}}|v_0(x)|. \tag{15}$$

For this, only the one-sided estimate (14) is required.

Differentiating equation (11) with respect to t, we find that

$$\begin{aligned} v''_{tt} - \left[a_{ik}v'_{x_k,t} + (a_{ik})'_t v'_{x_k}\right]_{x_i} + (a_{ik})'_{x_i,t}v'_{x_k} + (a_{i,k})_{x_i}v'_{x_k,t} + (a_i)'_t v'_{x_i} \\ + D_{x_i}(a_i)v'_t - (a_i)'_{x_i}v'_t + a_i v''_{x_i,t} + a'_t + a_v v'_t = 0. \end{aligned} \tag{16}$$

Here the symbol D_{x_i} denotes the total partial derivative with respect to x_i. Differentiation of the boundary conditions with respect to t gives

$$v'_t = 0 \quad \text{or} \quad (a_{ik})'_t v''_{x_k x_i}\cos(n,x_i) + a_{ik}v''_{x_k,t}\cos(n,x_i) +$$

$$+\sigma'_t v + \sigma'_v v'_t v + \sigma v'_t = 0. \tag{17}$$

Multiply both sides of identity (16) by $v'_t|v'_t|^{p-2}$ and integrate over Ω. In the second and sixth terms we perform integration by parts, taking into account the boundary conditions (17). We obtain an identity, which we denote by $(*)$.

From the positive definiteness of the form $a_{ik}\gamma_i\gamma_k$ it follows that

$$\int_\Omega a_{ik} v''_{x_k,t} (v'_t |v'_t|^{p-2})'_{x_i} \, dx \geq \lambda_p(T) \int_\Omega |v''_{x_i,t}|^2 |v'_t|^{p-2} \, dx. \tag{18}$$

Applying estimate (18) and Hölder' s inequality to the identity (∗), we obtain

$$\left[\int_\Omega |v'_t|^p \, dx\right]'_t \leq A_p(T) \int_\Omega |v'_t|^p \, dx + B_p(T). \tag{19}$$

Moreover, in order to estimate the integral $\int_\Omega |v'_{x_i}| \, |v'_t|^{p-1} \, dx$, one must take into account that

$$\|v'_{x_i}\|_{L_p} \leq \varepsilon \|v'_t\|_{L_p} + C(\varepsilon, T). \tag{20}$$

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* Here $A(t)$ is the operator defined by the differential expression $-a_{ik}(t,x) v''_{x_i x_k}$ and the boundary conditions (13).

Indeed, from (11), (13), and (15) it follows that

$$\|A(t)v\|_{L_p} \leqslant C_p(T) \|v'_t\|_{L_p} + D_p(T). \tag{21}$$

Applying (15) further, we obtain (20).

In the case when the second of the boundary conditions (17) is satisfied, it is also necessary to take into account the inequality

$$\int_S |v'_t|^p \, dx \leqslant \varepsilon \int_\Omega |v''_{x_i,t}|^2 |v'_t|^{p-2} \, dx + C(\varepsilon) \int_\Omega |v'_t|^p \, dx, \tag{22}$$

which follows from S. L. Sobolev' s embedding theorems and Hölder' s inequality.

From (19) it follows easily that

$$\|v'_t\|_{L_p} \leqslant E_p(T). \tag{23}$$

Hence, from (21), the required a priori estimate follows:

$$\|A(t)v\|_{L_p} \leqslant F_p(T). \tag{24}$$

6. Let us note that the a priori estimate (24) was obtained by us without a preliminary estimate of $\max_{x\in\overline{\Omega}} |v_{x_i}|$. Such estimates follow as a consequence of estimate (24) (item 3). The estimate of $\max_{x\in\overline{\Omega}} |v_{x_i}|$ is usually the most difficult stage of the proof. In doing so, one uses, for example, Schauder' s delicate constructions ([12]).

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Note: Figure translations are in progress. See original paper for figures.