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Abstract

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ON THE SOLUTION IN THE LARGE OF THE FIRST BOUNDARY-VALUE PROBLEM FOR A QUASILINEAR PARABOLIC EQUATION

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In the present paper we study the first boundary-value problem for a quasilinear parabolic equation of the form

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t,u)\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial t} = f(x,t,u,\nabla u), \quad (1)$$

where $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \dots, \partial u/\partial x_n)$, in noncylindrical domains D. A. Friedman studied an analogous problem (for $b_i(x,t,u) \equiv b_i(x,t)$), establishing in paper (1) an a priori $(1+\delta)$-estimate for the solution of the first boundary-value problem for a linear parabolic equation. However, under the restrictions imposed by A. Friedman on $f(x,t,u,\omega)$, the existence of a solution was proved locally (with respect to T). We shall consider the question of existence and uniqueness of the solution of the first boundary-value problem for equation (1) in the large, i.e., for arbitrary T.

Let D be an $(n+1)$-dimensional domain of the space $(x_1, x_2, \dots, x_n; t) \equiv (x,t)$, bounded by two hyperplanes $t = 0$ and $t = T > 0$ and by a closed surface S lying between these hyperplanes. Let Ω be the base of D, i.e. $\Omega = \overline{D} \cap \{t = 0\}$. We shall call the set $\Gamma = S \cup \Omega$ the normal boundary of the domain. Following A. Friedman (1), we introduce the following norms:

$$|v|_0^D = \sup_{(x,t)\in D} |v(x,t)|, \qquad |v|_\alpha^D = |v|_0^D + H_\alpha^D[v],$$

$$H_\alpha^D[v] = \sup_{P_1,P_2\in D} \frac{|v(P_1) - v(P_2)|}{[d(P_1,P_2)]^\alpha},$$

where the distance between two points $P_1(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n; \bar{t})$ and $P_2(\bar{\bar{x}}_1, \bar{\bar{x}}_2, \dots, \bar{\bar{x}}_n; \bar{\bar{t}})$ is defined by

$$d(P_1, P_2) = \left(\sum_{i=1}^{n} (\bar{x}_i - \bar{\bar{x}}_i)^2 + \left| \bar{t} - \bar{\bar{t}} \right| \right)^{1/2}. \tag{2}$$

Further,

$$|v|_{1+\alpha}^{D} = |v|_{\alpha}^{D} + \sum_{i=1}^{n} \left| \frac{\partial v}{\partial x_i} \right|_{\alpha}^{D},$$

$$|v|_{2+\alpha}^{D} = |v|_{1+\alpha}^{D} + \sum_{i=1}^{n} \left| \frac{\partial v}{\partial x_i} \right|_{1+\alpha}^{D} + \left| \frac{\partial v}{\partial t} \right|_{\alpha}^{D}.$$

I. With respect to the lateral surface S it is assumed that it can be covered by a finite number of spheres W_j, and in each sphere W_j po-

the piece of the surface S_j falling into it admits, for some i, a representation of the form

$$\begin{gathered} x_i = h(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n; t), \\ (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n, t) \in \Sigma_j, \end{gathered}$$

where the function h has on Σ_j derivatives with respect to x_k up to the second order inclusive, satisfying the Hölder condition (with exponent α; $0 < \alpha < 1$) and a first derivative with respect to t, also satisfying the Hölder condition (with exponent α); here the distance between the points $P_1(\bar{x}, \bar{t})$ and $P_2(\bar{\bar{x}}, \bar{\bar{t}})$ in the Hölder condition is taken from (2). In addition it is assumed that $\partial h / \partial x_k$ on Σ_j satisfies the Lipschitz condition with the usual distance

$$\rho(P_1, P_2) = \left(\sum_{i=1}^{n} (\bar{x}_i - \bar{\bar{x}}_i)^2 + (\bar{t} - \bar{\bar{t}})^2 \right)^{1/2}. \tag{3}$$

Let the quasilinear operator L in (1) be parabolic for $(x, t) \in \overline{D}$, i.e., for all real vectors $(\lambda_1, \lambda_2, \dots, \lambda_n)$

$$\sum_{i,j=1}^{n} a_{ij}(x, t) \lambda_i \lambda_j \geq a_0 \sum_{i=1}^{n} \lambda_i^2. \tag{4}$$

Let the coefficients and the right-hand side of equation (1) satisfy the conditions:

II. For all $(x, t) \in \overline{D}$, $|u| < \infty$, $\partial f(x, t, u, 0)/\partial u \geq b_0$ (b_0 is a constant).

III. In the domain $(x, t) \in \overline{D}$, $|w| < \infty$ $\left(|w|^2 = \sum_{i=1}^{n} w_i^2\right)$ and $|u| \leq K \equiv$

$$\equiv \left(\sup_{\Gamma} |\psi| + \frac{\sup |f(x,t,0,0)|}{b_0 + \gamma} \right) e^{\gamma T}$$

($\gamma > 0$ is a constant for which $\gamma + b_0 > 0$), the following conditions are fulfilled: the functions $a_{ij}(x,t), b_i(x,t,u), f(x,t,0,0), \partial f(x,t,u,0)/\partial u$, and $\partial f(x,t,u,w)/\partial w_i$ $(i = 1,2,\dots,n)$ satisfy in (x,t) the Hölder condition (with exponent α; $0 < \alpha < 1$), and in u and w_i $(i = 1,2,\dots,n)$ the Hölder conditions (with exponent β, $0 < \beta \leq 1$):

$$|a_{ij}(x,t) - a_{ij}(\bar{x},\bar{t})| \leq A_1 [d(P_1,P_2)]^{\alpha} \tag{5}$$

(here and below $P_1 = P_1(x,t)$; $P_2 = P_2(\bar{x},\bar{t})$ and $d(P_1,P_2)$ is taken from (2));

$$|b_i(x,t,u) - b_i(\bar{x},\bar{t},\bar{u})| \leq B_1 [d(P_1,P_2)]^{\alpha} + B_2 |u - \bar{u}|^{\beta}, \tag{6}$$

$$|f(x,t,0,0) - f(\bar{x},\bar{t},0,0)| \leq C_1 [d(P_1,P_2)]^{\alpha},$$

$$\left| \frac{\partial f(x,t,u,0)}{\partial u} - \frac{\partial f(\bar{x},\bar{t},\bar{u},0)}{\partial u} \right| \leq C_2 [d(P_1,P_2)]^{\alpha} + C_3 |u - \bar{u}|^{\beta}; \tag{7}$$

$$\left| \frac{\partial f(x,t,u,w)}{\partial w_i} \right| \leq C_4 \qquad (i = 1,2,\dots,n);$$

$$\left| \frac{\partial f(x,t,u,w)}{\partial w_i} - \frac{\partial f(\bar{x},\bar{t},\bar{u},\bar{w})}{\partial w_i} \right| \leq D_1 [d(P_1,P_2)]^{\alpha} +$$

$$+ D_2 |u - \bar{u}|^{\beta} + D_3 \left[\sum_{i=1}^{n} (w_i - \bar{w}_i)^2 \right]^{\beta/2} \qquad (i = 1,2,\dots,n). \tag{8}$$

IV. The functions $a_{ij}(x,t)$ on Σ_j satisfy, with respect to (x,t), the Lipschitz condition with the usual distance (3).

V. Let there exist in the domain $\overline{D}$ a function $\Psi(x,t)$, coinciding on the normal boundary Γ of the domain D with the prescribed boundary function $\psi(x,t)$, and such that $|\Psi|_{2+\alpha}^{D} < \infty$.

Theorem 1. *If the lateral surface S, the coefficients of the quasilinear parabolic equation (1), and the boundary function $\psi(x,t)$ satisfy all the conditions (4), I $-$V indicated above, then there exists a solution $u(x,t)$, continuous in $\overline{D}$, of equation (1) with the prescribed boundary conditions*

$$u|_{\Gamma} = \psi(x,t), \tag{9}$$

and constants M and λ $(0 < \lambda \leq \alpha\beta < 1)$ can be found such that in $\overline{D}$ the inequality

$$|u|^D_{2+\lambda} \leq M(|f(x,t,0,0)|_\alpha + |\Psi|_{2+\alpha}), \tag{10}$$

holds, where M depends on the domain D, the surface S, and the constants $\alpha, \beta, \lambda, K, a_0, A_1, B_1, B_2, C_1, C_2, C_3, C_4, D_1, D_2, D_3$.

The proof of Theorem 1 uses an a priori estimate for the modulus of a solution of the problem (1), (9), A. Friedman' s theorem on the $(1+\delta)$-estimate for a solution of the first boundary-value problem for a linear parabolic equation (1), A. Friedman' s theorem on the existence and the $(2+\alpha)$-estimate for a solution of the first boundary-value problem for a linear parabolic equation (2), and Schauder' s fixed-point theorem.

Theorem 2 (A. Friedman ([1])). *Let S be any closed surface and let the quasilinear operator L be parabolic in $\overline{D}$. If $b_i(x,t,u)$ and $f(x,t,u,w)$ are locally Lipschitz continuous with respect to u, i.e., for $|u| \leq N$*

$$|b_i(x,t,u_1) - b_i(x,t,u_2)| \leq B_2(N)|u_1 - u_2|,$$

$$|f(x,t,u_1,w) - f(x,t,u_2,w)| \leq C_2(N)|u_1 - u_2|,$$

then in D there can exist at most one solution of the first boundary-value problem (1), (9), continuous in $\overline{D}$.

Remark. From Theorem 2, for $\beta = 1$ it follows that the solution of the first boundary-value problem (1), (9), whose existence was proved in Theorem 1, is unique.

Let us note one generalization of Theorem 2 on uniqueness for a more general quasilinear parabolic equation.

Theorem 3. *Let S be any closed surface and let the quasilinear operator*

$$\Lambda u = \sum_{i,j=1}^{n} a_{ij}(x,t,u,\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t}$$

be parabolic in $\overline{D}$, i.e., for $(x,t) \in \overline{D}$

$$\sum_{i,j=1}^{n} a_{ij}(x,t,u,w)\lambda_i\lambda_j \geq a(u,w)\sum_{i=1}^{n}\lambda_i^2, \tag{11}$$

where $a(u,w) > 0$ is a nonincreasing function of $(|u| + |w|)$ for $(|u| + |w|) < \infty$.

If $a_{ij}(x,t,u,w)$ and $f(x,t,u,w)$ are locally Lipschitz continuous with respect to u, then there can exist at most one solution of the first boundary-value problem for the equation

$$\Lambda u \equiv f(x,t,u,\nabla u) \tag{12}$$

with boundary condition (9), continuous in $\overline{D}$ and having in $\overline{D}$ bounded derivatives $\partial u/\partial x_i$, $\partial^2 u/\partial x_i \partial x_j$ $(i,j = 1,2,\dots,n)$.

With the aid of Lemma 1, Theorem 4 is proved.

Lemma 1. If, for the function $f(x,t,u,w)$, continuous in all arguments for $|u| < \infty$, the inequality

$$|f(x,t,u,0)| \leq C_5 + C_6|u|, \tag{13}$$

holds, then for every solution, continuous in $\overline{D}$, of problem (12), (9) (where Λ from (12) with continuous coefficients $a_{ij}(x,t,u,\nabla u)$ satisfies (11)), the following a priori estimate holds:

$$\sup_{\overline{D}} |u(x,t)| \leq K_1 \equiv \left(\sup_{\Gamma} |\psi| + \frac{C_5}{\gamma - C_6}\right) e^{\gamma T} \tag{14}$$

($\gamma > 0$ is any constant for which $\gamma - C_6 > 0$).

Theorem 4. Let the lateral surface S satisfy condition I; let the boundary function $\psi(x,t)$ satisfy condition V; let $a_{ij}(x,t)$, continuous in the Hölder sense (5), satisfy in $\overline{D}$ the parabolicity condition (4), as well as condition IV. Suppose that for all $|u| < \infty$ (13) is fulfilled, and that in the domain $(x,t) \in \overline{D}$, $|w| < \infty$, $|u| \leq K_1$ (where the constant K_1 is taken from (14)) the functions $b_i(x,t,u)$, $f(x,t,u,0)$, $\partial f(x,t,u,w)/\partial w_i$ are Hölder continuous, i.e. (6), (8) are fulfilled, and

$$|f(x,t,u,0) - f(\bar{x},\bar{t},\bar{u},0)| \leq C_7[d(P_1,P_2)]^\alpha + C_8|u-\bar{u}|^\beta,$$

and for $\partial f(x,t,u,w)/\partial w_i$ inequality (7) holds. Then there exists a solution $u(x,t)$, continuous in $\overline{D}$, of the first boundary-value problem (1), (9), for which (10) is valid, where M depends on the domain D, the surface S, and on $\alpha, \beta, \lambda, A_1, B_1, B_2, C_4, C_5, C_6, C_7, C_8, D_1, D_2, D_3$.

Remark. In A. Friedman' s paper (^{1}), for equation (1) ($b_i(x,t,u) \equiv b_i(x,t)$) it is proved that: 1) if one does not require the existence of $\partial f/\partial w_i$, but imposes the condition

$$|f(x,t,u,w)| \leq C_5 + C_6|u| + C_9|w|^\delta, \tag{15}$$

where $0 \leq \delta < 1$, then the solution of problem (1), (9) exists globally; 2) if in (15) $\delta = 1$ and one does not require the existence of $\partial f / \partial w_i$, then the solution of problem (1), (9) exists for small C_9; 3) if $f(x, t, u, w)$ is locally Hölder continuous, then the solution of problem (1), (9) exists locally (with respect to T).

Note added in proof. The existence of a solution of problem (1), (9) was proved by the authors for a lateral surface S admitting a barrier at each point, and for a boundary function ψ continuous on Γ.

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REFERENCES

1. A. Friedman, *J. Math. and Mech.*, **9**, No. 4, 539 (1960).
2. A. Friedman, *J. Math. and Mech.*, **7**, No. 5, 771 (1958).

Note: Figure translations are in progress. See original paper for figures.

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