# ON AN ESTIMATE OF THE MIXED DERIVATIVE IN \(L_p(G)\)

1961

SovietRxiv

## Abstract

## Full Text

## MATHEMATICS

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# ON AN ESTIMATE OF THE MIXED DERIVATIVE IN $L_p(G)$

*(Presented by Academician P. S. Novikov, 3 VI 1961)*

1. Let $f(x_1, x_2)$ in a domain $G$ have generalized unmixed derivatives $\partial^2 f/\partial x_1^2$, $\partial^2 f/\partial x_2^2$. We shall say that $f \in W_p^{(2,2)}(G)$ if

$$\|f\|_{W_p^{(2,2)}} = \left\{ \|f\|_{L_p(G)}^p + \sum_1^2 \left\| \frac{\partial f}{\partial x_i} \right\|_{L_p(G)}^p + \sum_1^2 \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_{L_p(G)}^p \right\}^{1/p} < \infty. \quad (1)$$

From the results of S. M. Nikol' skii ([1–3]) on the extension of functions and of S. G. Mikhlin ([8,9]) it follows that the function $f \in W_p^{(2,2)}(G)$ has a generalized mixed derivative $\partial^2 f/\partial x_1 \partial x_2$ in $G$, and moreover, for $1 < p < \infty$,

$$\left\| \frac{\partial^2 f}{\partial x_1\, \partial x_2} \right\|_{L_p(G_\varepsilon)} \le C \|f\|_{W_p^{(2,2)}(G)}, \quad (2)$$

where $G_\varepsilon$ is the domain internal to $G$, consisting of those points of $G$ whose distances to the boundary of $G$ are greater than $\varepsilon > 0$, and the constant $C$ does not depend on $f$, but depends on the domain and on $\varepsilon$.

2. In the following particular cases it is known that in inequality (2) one may replace $G_\varepsilon$ by $G$:

$$\left\| \frac{\partial^2 f}{\partial x_1\, \partial x_2} \right\|_{L_p(G)} \le C \|f\|_{W_p^{(2,2)}(G)}. \quad (3)$$

1) The case of a periodic function; $p = 2$ (S. N. Bernstein ([4]), p. 97); for $1 < p < \infty$, (3) follows from the theorem of I. Marcinkiewicz ([5]) on multipliers of Fourier series.

2) The domain $G$ is a rectangle, finite or infinite, with sides parallel to the coordinate axes (L. N. Slobodetskii ([6,7]); for the case of the plane $G \equiv R_2$ it follows from the works ([8,9]) of S. G. Mikhlin).

3) The function $f$ is equal to zero on the boundary of the domain $G$, bounded by a curve of class $C^3$; $p = 2$ (S. N. Bernstein ([10]), S. G. Mikhlin ([12])); $1 < p < \infty$ (A. I. Koshelev ([13])).

4) The boundary values of the function and of its normal derivative belong to the classes:

$$f|_\Gamma \in W_p^{(2-1/p)}(\Gamma), \qquad \left.\frac{\partial f}{\partial n}\right|_\Gamma \in W_p^{(1-1/p)}(\Gamma), \qquad 1 < p < \infty$$

(L. N. Slobodetskii ([10])).

In the case $p = \infty$ (the metric $C$) estimate (3) does not hold (B. S. Mityagin ([11])).

3. We have established that inequality (3) is valid for an arbitrary domain $G$ with twice continuously differentiable boundary $\Gamma$, without any assumptions about boundary values. Namely:

**Theorem 1.** *Let the domain $G$ be bounded by a finite number of closed or unbounded curves $\Gamma_i$ of class $C^2$,*

$$\min_{i \neq k} \rho(\Gamma_i, \Gamma_k) > 0, \qquad 1 < p < \infty; \qquad f \in W_p^{(2,2)}(G).$$

Then the mixed derivative is summable in the $p$-th power in the domain $G$ and the inequality holds

$$\left\| \frac{\partial^2 f}{\partial x_1 \partial x_2} \right\|_{L_p(G)} \leq C \|f\|_{W_p^{(2,2)}(G)}, \tag{4}$$

where the constant $C$ does not depend on $f$.

The proof is based on lemmas, proved by us, concerning the properties of boundary values of the function $f$ and its derivatives.

**Lemma 1.** Let the domain $G$ contain the strip $a < x_1 < b$, $\alpha(x_1) < x_2 < \alpha(x_1) + d$, where the curve $x_2 = \alpha(x_1)$ is a part $\gamma$ of the boundary $\Gamma$ of the domain $G$, and

$$\int_a^b |\alpha''(t)|^p \, dt < \infty, \qquad |\alpha'(x_1)| \leq k < 1.$$

Let $f(x_1, x_2) \in W_p^{(2,2)}(G)$, $1 < p < \infty$. Let

$$\varphi(x_1) = f\big|_\gamma, \qquad \mu(x_1) = \left.\frac{\partial f}{\partial x_2}\right|_\gamma.$$

Then

$$\varphi(x_1) \in W_p^{(2-1/p)}(a_1, b_1), \qquad \mu(x_1) \in W_p^{(1-1/p)}(a_1, b_1),$$

$$\|\varphi\|_{W_p^{(2-1/p)}(a_1,b_1)} + \left\|\frac{d\varphi}{dx_1}\right\|_{W_p^{(1-1/p)}(a_1,b_1)} + \|\mu\|_{W_p^{(1-1/p)}(a_1,b_1)} \le C\|f\|_{W_p^{(2,2)}(G)}, \quad (5)$$

where $(a_1, b_1)$ is an arbitrary interval lying inside $(a, b)$.

**Lemma 2.** Let the domain $G$ contain the strip $a < x_1 < b$, $\alpha(x_1) < x_2 < \alpha(x_1) + d$, where the curve $x_2 = \alpha(x_1)$ is a part $\gamma$ of the boundary $\Gamma$ of the domain $G$, and

$$\int_a^b |\alpha''(t)|^p \, dt < \infty, \qquad 0 < k_1 \le \alpha'(x_1) \le k_2 < \infty.$$

Let $f \in W_p^{(2,2)}(G)$, $1 < p < \infty$,

$$\varphi(x_1) = f\big|_\gamma, \qquad \lambda(x_1) = \frac{\partial f}{\partial x_1}\bigg|_\gamma, \qquad \mu(x_1) = \frac{\partial f}{\partial x_2}\bigg|_\gamma.$$

Then

$$\varphi(x_1) \in W_p^{(2-1/p)}(a_1, b_1), \qquad \lambda(x_1), \mu(x_1) \in W_p^{(1-1/p)}(a_1, b_1),$$

$$\|\varphi\|_{W_p^{(2-1/p)}(a_1,b_1)} + \left\|\frac{d\varphi}{dx_1}\right\|_{W_p^{(1-1/p)}(a_2,b_1)} + \|\lambda\|_{W_p^{(1-1/p)}(a_1,b_1)} +$$

$$+ \|\mu\|_{W_p^{(1-1/p)}(a_1,b_1)} \le C\|f\|_{W_p^{(2,2)}(G)}, \quad (6)$$

where $(a_1, b_1)$ is an arbitrary interval lying inside $(a, b)$.

**Lemma 3.** Almost everywhere on the set $E$ of those points of the boundary $\Gamma$ of the domain $G$ at which the tangent to $\Gamma$ is not parallel to either of the coordinate axes, the formula holds

$$\varphi'(x_1) = \lambda(x_1) + \mu(x_1)\alpha'(x_1),$$

where $x_2 = \alpha(x_1)$ is the equation of an arc of the boundary.

Analogous circumstances hold for $n$ variables.

**Theorem 2.** Let

$$\|f\|_{W_p^{(2,\dots,2)}(G)} = \left\{ \|f\|_{L_p(G)}^p + \sum_1^n \left\| \frac{\partial f}{\partial x_i} \right\|_{L_p(G)}^p + \sum_1^n \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_{L_p(G)}^p \right\}^{1/p} < \infty \quad (7)$$

and let the domain $G$ be bounded by a finite number of surfaces $\Gamma_i$ of class $C^2$,

$$\min_{i \neq k} \rho(\Gamma_i, \Gamma_k) > 0.$$

Then the mixed derivatives are summable to the $p$-th power on the domain $G$, and

$$\left\| \frac{\partial^2 f}{\partial x_i\, \partial x_k} \right\|_{L_p(G)} \leq C\, \|f\|_{W_p^{(2,\dots,2)}(G)}. \quad (8)$$

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Received
25 V 1961

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*Note: Figure translations are in progress. See original paper for figures.*