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# MATHEMATICS

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**Abstract**

**Full Text**

## MATHEMATICS

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### A GENERATING FUNCTION FOR THE CLEBSCH-GORDAN COEFFICIENTS

*(Presented by Academician A. N. Kolmogorov, January 21, 1961)*

In the present note a generating function is constructed for the Clebsch-Gordan coefficients, which makes it possible, in the study of these coefficients, to apply methods analogous to those used in the theory of special functions.

The Clebsch-Gordan coefficients  $C(l_1 l_2 m_1 m_2 | lm)$  <sup>(1)</sup> are usually defined as the coefficients in the expansion

$$g_m^l = \sum_{m_1+m_2=m} C(l_1 l_2 m_1 m_2 | lm) e_{m_1}^{l_1} f_{m_2}^{l_2}, \quad (1)$$

where  $g_m^l$  ( $|l_1 - l_2| \leq l \leq l_1 + l_2$ ) are vectors of the canonical bases <sup>(2)</sup> in the space of the Kronecker product of two irreducible representations of weights  $l_1$  and  $l_2$  of the three-dimensional rotation group;  $e_{m_1}^{l_1}$  and  $f_{m_2}^{l_2}$  ( $-l_1 \leq m_1 \leq l_1$ ,  $-l_2 \leq m_2 \leq l_2$ ) are the canonical bases of the Kronecker "factors."

When the irreducible representations are realized in the spaces of polynomials  $p(z_1)$  and  $p(z_2)$  of degrees not exceeding respectively  $2l_1$  and  $2l_2$ , by the operators

$$T_\varphi p(z) = e^{-il\varphi} p(e^{i\varphi} z),$$

$$T_\vartheta p(z) = \left( iz \sin \frac{\vartheta}{2} + \cos \frac{\vartheta}{2} \right)^{2l} p \left( \frac{z \cos \frac{\vartheta}{2} + i \sin \frac{\vartheta}{2}}{iz \sin \frac{\vartheta}{2} + \cos \frac{\vartheta}{2}} \right),$$

the canonical bases of the representations have the form

$$e_{m_1}^{l_1} = h_{m_1}^{l_1} z_1^{l_1 - m_1}, \quad f_{m_2}^{l_2} = h_{m_2}^{l_2} z_2^{l_2 - m_2}, \quad h_m^l = \frac{(-1)^{l-m}}{\sqrt{(l-m)!(l+m)!}}, \quad (2)$$

where  $\varphi, \vartheta$  are Euler angles; the Kronecker product is specified in the space of polynomials  $p(z_1, z_2)$  in two variables, of degree not exceeding  $2l_1$  in the first and  $2l_2$  in the second variable.

This Kronecker product decomposes into a direct sum of irreducible representations. The polynomials  $p_m^l$ , which serve as canonical bases of the named representations, can be found with the aid of the infinitesimal operators  $H_-, H_+, H_3$  (3):

$$H_- = z_1^2 \frac{\partial}{\partial z_1} + z_2^2 \frac{\partial}{\partial z_2} - 2(l_1 z_1 + l_2 z_2),$$

$$H_+ = -\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2}, \quad H_3 = -z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} + (l_1 + l_2).$$

Taking into account, together with (2), the action of the infinitesimal operators on the vectors of the canonical basis:

$$H_- p_m^l = \alpha_m^l p_{m-1}^l, \quad H_+ p_m^l = \alpha_{m+1}^l p_{m+1}^l,$$

$$H_3 p_m^l = m p_m^l, \quad \alpha_m^l = \sqrt{(l+m)(l-m+1)},$$

the polynomials  $p_m^l(z_1, z_2)$  can be represented in the form

$$p_m^l(z_1, z_2) = 2^{-(l_1+l_2)} z_1^{\frac{l_1-l_2-m}{2}} z_2^{\frac{l_2-l_1-m}{2}} q_m^l(\mu), \quad \mu = \frac{z_1 + z_2}{z_1 - z_2},$$

where  $q_m^l(\mu)$  satisfies the well-known recurrence relation

$$\sqrt{1-\mu^2} \frac{dq_m^l(\mu)}{d\mu} + \frac{m\mu - (l_2 - l_1)}{\sqrt{1-\mu^2}} q_m^l = -i\alpha_{m+1}^l q_{m+1}^l$$

between the "generalized spherical functions"  $P_{mn}(\mu)$  (2).

Hence, since  $q_{-l}^l(\mu)$  coincides, up to a constant factor, with the function  $P_{-l, l_2-l_1}^l(\mu)$ ,

$$p_m^l(z_1, z_2) = (-1)^{l-m} a_l z_1^{\frac{l_1-l_2-m}{2}} z_2^{\frac{l_2-l_1-m}{2}} P_{l_2-l_1, m}^l \left( \frac{z_1 + z_2}{z_1 - z_2} \right), \quad (3)$$

$$a_l = \sqrt{\frac{2l+1}{(l_1+l_2-l)!(l_1+l_2+l+1)!}}$$

Substituting  $e_{m_1}^{l_1}, f_{m_2}^{l_2}$  from (2) and  $g_m^l = p_m^l$  from (3) into (1), after cancellation by  $z_1^{l_1+l_2-m}$ , we obtain the expansion

$$\begin{aligned} & (-1)^{l-m} a_l t^{\frac{l_2-l_1-m}{2}} (1-t)^{l_1+l_2} P_{l_2-l_1, m}^l \left( \frac{1+t}{1-t} \right) = \\ & = \sum_{m_1+m_2=m} C(l_1 l_2 m_1 m_2 | lm) n_{m_1}^{l_1} n_{m_2}^{l_2} t^{l_2-m_2}, \quad t = \frac{z_2}{z_1}, \end{aligned}$$

whose left-hand side may be regarded as a “generating function” for the Clebsch-Gordan coefficients. The latter may also be written in other forms, for example through the hypergeometric function:

$$\begin{aligned} & (-1)^{l-m} a_l \frac{1}{(m+l_1-l_2)!} \sqrt{\frac{(l+l_1-l_2)!(l+m)!}{(l+l_2-l_1)!(l-m)!}} (1-t)^{l_1+l_2-l} \times \\ & \quad \times F(m-l, l_1-l_2-l, l_1-l_2+m+1, t) = \\ & = \sum_{m_1+m_2=m} C(l_1 l_2 m_1 m_2 | lm) n_{m_1}^{l_1} n_{m_2}^{l_2} t^{l_2-m_2} \end{aligned}$$

for  $l_2 - l_1 \leq m \leq l$ , and a similar expression for  $-l \leq m \leq l_2 - l_1$ .

Using the generating function and the unitarity property of the matrix  $\|C(l_1 l_2 m_1 m_2 | lm)\|$ , it is not difficult to write the Clebsch-Gordan coefficients in forms analogous to Rodrigues’ formula and to the integral representation of special functions:

$$\begin{aligned} C(l_1 l_2 m_1 m_2 | lm) &= (-1)^{l-m} a_l \left[ n_{m_1}^{l_1} n_{m_2}^{l_2} (l_2 - m_2)! (m + l_1 - l_2)! \right]^{-1} \times \\ & \quad \times \sqrt{\frac{(l+l_1-l_2)!(l+m)!}{(l+l_2-l_1)!(l-m)!}} \times \\ & \quad \times \left\{ \frac{d^{l_2-m_2}}{dt^{l_2-m_2}} (1-t)^{l_1+l_2-l} F(m-l, l_1-l_2-l, l_1-l_2+m+1, t) \right\}_{t=0}, \\ C(l_1 l_2 m_1 m_2 | ml) &= (-1)^{l-m} i^{l_1+l_2+m_1-m_2} 2^{-(l_1+l_2)} \frac{(2l+1) n_{m_1}^{l_1} n_{m_2}^{l_2}}{a_l} \times \\ & \quad \times \int_{-1}^{+1} (1-\mu^2)^{\frac{l_1+l_2}{2}} \left( \frac{1-\mu}{1+\mu} \right)^{\frac{m_1-m_2}{2}} \bar{P}_{l_1-l_2, m}^l(\mu) d\mu, \end{aligned}$$

and also obtain for them the commonly used Wigner sum (1) and various properties, both known and new. The latter are derived simply by using the properties of hypergeometric and generalized spherical functions.

As an example, we indicate an interesting asymptotic expression for the Clebsch-Gordan coefficients

$$C(l_1 l_2 m_1 m_2 | lm) \approx (-1)^{l_1+l_2-l} \frac{(l+m+1)^{l_2-m_2}}{(l-l_2-m_1)!} \times \\ \times \sqrt{\frac{(2l+1)(l+l_1-l_2)!(l+m)!(l-m)!(l_1-m_1)!(l_2+m_2)!}{(l+l_2-l_1)!(l_1+l_2-l)!(l_1+l_2+l+1)!(l_1+m_1)!(l_2-m_2)!}}$$

valid for  $l_1, l_2, l, m \gg l_1 + l_2 - m$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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