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Abstract

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METRIC PROPERTIES OF ENDOMORPHISMS OF THE n-DIMENSIONAL TORUS

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It is known that any group endomorphism of the n-dimensional torus can be regarded as a metric endomorphism of the torus ([3]). In the present paper some properties of metric endomorphisms of the n-dimensional torus are established.

Let E^n be n-dimensional Euclidean space in which an orthogonal normalized basis has been chosen. It is known that E^n is an Abelian group under addition, and the set of points of E^n with integral coordinates is a subgroup, which we shall denote by Δ^n. The factor group E^n/Δ^n is a certain torus T^n. A group endomorphism S^n of the torus T^n is generated by a certain linear transformation $\widetilde{S}_n$ of the space E^n, which is written in the basis chosen by us as an integral matrix. The following theorem reduces the study of an arbitrary endomorphism to the study, in a known sense, of "elementary" endomorphisms.

Theorem 1. Let

$$p(\lambda) = p_1^{n_1}(\lambda) \cdot \ldots \cdot p_m^{n_m}(\lambda)$$

be the decomposition of the characteristic polynomial of the matrix $\widetilde{S}_n$ into factors irreducible over the field of rational numbers. Then the torus T^n can be decomposed into a direct product of tori T_i, where the dimension of the i-th torus is $n_i k_i$, where k_i is the degree of the polynomial $p_i(\lambda)$, $i = 1, \ldots, m$. Each torus T_i is invariant with respect to the endomorphism S_n, and if the endomorphism acting on the torus T_i is denoted by S_i, then the endomorphism S_n admits the direct decomposition

$$S_n = S_1 \times S_2 \times \ldots \times S_m.$$

Proof. Let $\lambda_{11}, \ldots, \lambda_{1k_1}$ be the roots of the polynomial $p_1(\lambda)$. From the irreducibility of the polynomial it can be inferred that they are either all real, or all complex (with nonzero imaginary parts).

Consider the case when all roots λ_{1i} are real. Then the linear transformation $\widetilde{S}_n$ has an eigenvector $\tilde{x}_{11}^{(1)}$

$$\widetilde{S}_n \tilde{x}_{11}^{(1)} = \lambda_{11} \tilde{x}_{11}^{(1)}.$$

The set of vectors $(\{t\tilde{x}_{11}^{(1)}\}, -\infty < t < \infty)$ forms a subgroup in E^n. The closure of the image of this group in T^n is a closed connected subgroup of the torus T^n, i.e. a certain torus of dimension k. By a well-known theorem of Kronecker ([7]), the number k is equal to the number of rationally independent coordinates of the vector $\tilde{x}_{11}^{(1)}$. These coordinates are certain polynomials in λ_{11} with integral coefficients; from the irreducibility of the polynomial $p_1(\lambda)$ it follows that there exist no more than k_1 such rationally independent polynomials, i.e.

$$k \leq k_1. \tag{1}$$

The constructed torus T^k is invariant with respect to S_n. Consequently, its complete inverse image E^k in the space E^n is invariant with respect to $\widetilde{S}_n$. But E^k

there is a certain k-dimensional linear subspace, defined by $n-k$ linear equations with integer coefficients. In it one can choose a basis so that the transformation $\widetilde{S}_n$ in E^k is given by an integral matrix, and the vector $\tilde{x}_{11}^{(1)}$ will belong to E^k. Therefore $\lambda_{11}^{(1)}$ will be an eigenvalue for the transformation $\widetilde{S}_n$ on E^k, and hence will satisfy the characteristic polynomial of the transformation $\widetilde{S}_n$ on E^k. Since the degree of this polynomial is equal to k, we obtain

$$k \geq k_1. \tag{2}$$

From comparison of (1) and (2) it follows that $k = k_1$. Therefore the characteristic polynomial of $\widetilde{S}_n$ on E^k coincides with $p_1(\lambda)$. The root λ_{11} has multiplicity n_1, and then either there exists another eigenvector $\tilde{x}_{11}^{(2)}$ with the same eigenvalue λ_{11}, in which case the preceding construction is applicable, or there exists a vector $\tilde{x}_{11}^{(2)}$ satisfying the equation

$$\widetilde{S}_n \tilde{x}_{11}^{(2)} = \lambda_{11} \tilde{x}_{11}^{(2)} + \tilde{x}_{11}^{(1)}.$$

We construct, from the vector $\tilde{x}_{11}^{(2)}$, the torus $T_1^{k_1}$ and the subspace $E_1^{k_1}$ analogously to the torus T^{k_1} and the subspace E^{k_1}. It is easy to see that $T^{2k_1} = T^{k_1} \times T_1^{k_1}$ is a torus invariant with respect to the endomorphism S_n, and the characteristic polynomial of the transformation $\widetilde{S}_n$ on $E^{k_1} \times E_1^{k_1}$ is $p_1^2(\lambda)$. Carrying out such a construction with all the vectors $\tilde{x}_{11}^{(i)}$, we obtain a torus $T^{n_1 k_1}$ invariant with respect to S_n and a Euclidean space $E^{n_1 k_1}$ invariant with respect to $\widetilde{S}_n$. The characteristic polynomial of $\widetilde{S}_n$ on $E^{n_1 k_1}$ is $p_1^{n_1}(\lambda)$. Constructing tori for all polynomials $p_i^{n_i}(\lambda)$, we obtain the required decomposition. The case of imaginary roots differs in no way from the one considered. The theorem is proved.

Let us recall the definitions (see ([1,4,5])).

Definition 1. A metric automorphism T of a Lebesgue space M is called a **Kolmogorov automorphism** if there exists a measurable partition ξ with the following properties:

1) $\xi < T\xi$; 2) $\prod_{n>0}^{\infty} T^n\xi = \varepsilon \bmod 0$, ε is the partition into individual points;

2) $\bigcup_{n=0}^{-\infty} T^n\xi = \nu \bmod 0$, ν is the trivial partition, whose only element is all of M.

Definition 2. A metric endomorphism T of a Lebesgue space M is called an **exact endomorphism** if $\bigcap_{n=0}^{\infty} T^{-n}\varepsilon = \nu$.

Theorem 2. If the characteristic polynomial $p(\lambda)$ of an endomorphism S_n of the torus T^n is representable in the form $p(\lambda) = \varphi^s(\lambda)$, where s is an integer and the polynomial $\varphi(\lambda)$ is irreducible, then S_n can be either an exact endomorphism or an automorphism.

For the proof, consider the endomorphism S_n on the torus T^k (k is the degree of the polynomial $\varphi(\lambda)$). If S_n is not an automorphism on T^k, then the preimage of 0 consists of several elements. The closure of the preimages of 0 under all S_n^p $(p = 1, 2, ...)$ is a torus T^{k_1} invariant with respect to S_n. If it does not coincide with T^k, then the characteristic polynomial of $\widetilde{S}_n$ on E^{k_1} divides $\varphi(\lambda)$, which is impossible by virtue of its irreducibility. Thus we obtain $T^k = T^{k_1}$, i.e., the preimages of 0 are dense in the torus T^k, but then they are dense also in T^n; hence it already follows easily that the endomorphism S_n is exact.

Corollary. An arbitrary endomorphism is the direct product of an exact endomorphism and an automorphism.

Theorem 3. If the characteristic polynomial of an ergodic automorphism S_n of the torus T^n has the form $p(\lambda) = \varphi^s(\lambda)$, where $\varphi(\lambda)$ is an irreducible polynomial, then S_n is a Kolmogorov automorphism.

We indicate the main points of the proof. From the ergodicity of the automorphism it follows that there exists a root λ_1 of the polynomial $p(\lambda)$ whose modulus is less than one. Otherwise all roots would have modulus one and the automorphism S_n would be nonergodic (see ([3],[8])).

Let the vectors $\tilde{x}_1, ..., \tilde{x}_k$ be eigenvectors for the transformation $\tilde{S}_n$ with eigenvalue λ_1, and let the vectors $\tilde{x}_{k+1}, ..., \tilde{x}_s$ satisfy the equation

$$\tilde{S}_n \tilde{x}_i = \lambda_1 \tilde{x}_i + \tilde{x}_{i-1}.$$

Represent the torus T^n as the Euclidean space E^n, in which points whose coordinates differ by an integer are identified. Denote by $\tilde{\xi}_1$ the partition of E^n into hyperplanes parallel to the linear subspace generated by the vectors $\tilde{x}_1, ..., \tilde{x}_s$, and by $\tilde{\xi}_2$ the partition into unit cubes with vertices at integral points. Denote the product of these partitions by $\tilde{\xi}_3$, and construct the partition $\tilde{\xi} = \tilde{\xi}_3 \vee \tilde{S}^{-1}\tilde{\xi}_3 \vee \ldots$. Denote by ξ the image of the partition $\tilde{\xi}$ in the torus T^n. It turns out that ξ has the following properties: 1) $S_n^k \xi < S_n^{k+1} \xi$; 2) $\bigvee_{k=0}^{\infty} S_n^k \xi = \varepsilon$; 3) $\bigwedge_{k=0}^{-\infty} S_n^k \xi = \nu$, i.e. S_n is a Kolmogorov automorphism.

Corollary. Every ergodic automorphism of the torus is a Kolmogorov automorphism.

This follows from the decomposition theorem and from the preceding theorem.

In conclusion we formulate a theorem which, in a special case, was proved by Ya. G. Sinai ([6]).

Theorem 4. The entropy of an ergodic automorphism of the torus is equal to

$$\sum_{|\lambda_i|>1} \log |\lambda_i|.$$

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Note: Figure translations are in progress. See original paper for figures.