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Abstract

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MECHANICS

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ON THE ROTATION OF A DEFORMABLE BODY

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§ 1. Consider the motion of a body (solid or liquid) whose center of mass is at rest in some inertial reference frame $Ox_1x_2x_3$, the origin of which we shall place at the center of mass. Suppose there is a "natural configuration" of the points of the body, in which their position is determined by the radius vector $\varrho = a_i\mathbf{e}_i$, where $\mathbf{e}_i$ are unit vectors along the axes $Ox_1x_2x_3$ (summation over a repeated index is understood). The displacements of the points (in general, finite), measured from the natural configuration, are denoted by the vector

$$\mathbf{u} = u_i(a_1, a_2, a_3; t)\mathbf{e}_i \tag{1,1}$$

(the Lagrangian method of describing the motion of a continuous medium); $u_{t0} = u_i(a_1, a_2, a_3; 0)$ is the initial displacement. The radius vector $\mathbf{r} = x_i(a_1, a_2, a_3; t)\mathbf{e}_i$ determines at the time t the position of the point which in the natural configuration had coordinates a_i. Then $x_i = a_i + u_i$, and $x_{i0} = a_i + u_{i0}$. We shall consider a finite linear deformation, i.e. we assume the function (1,1) to be linear; then

$$x_i = a_i + \frac{\partial u_i}{\partial a_k} a_k = a_i + (e_{ik} + \varphi_{ik})a_k. \tag{1,2}$$

The quantities e_{ik}, φ_{ik} (in general, finite) are assumed to have the same values throughout the body and are taken to be continuous and single-valued functions of time:

$$e_{ik} \equiv \frac{1}{2}(u_{ik} + u_{ki}) = e_{ki}(t); \quad \varphi_{ik} \equiv \frac{1}{2}(u_{ik}) - u_{ki} = -\varphi_{ki}(t); \quad u_{ik} \equiv \partial u_i/\partial a_k. \tag{1,3}$$

We call the body under consideration a body of homogeneous deformation, defined by equations (1,2) and (1,3). This body, whose points in the natural configuration are bounded by some continuous surface $f(a_1, a_2, a_3) = 0$, constitutes a mechanical system with 9 degrees of freedom (since its center of mass is fixed). As generalized coordinates of the system one may choose the 6 independent quantities e_{ik} and the three φ_{ik}. As is known[1], under a rotation of the coordinate axes the quantities e_{ik} transform as the components of a tensor of the second rank, and φ_{ik} as the components of an axial vector.

The deformation of the body itself is determined by the components of deformation ε_{ik} (in general, finite), defined by the equality

$$x_i x_i - a_i a_i = \varepsilon_{ik} a_i a_k. \tag{1,4}$$

From definitions (1,4) and (1,3) we obtain the expressions

$$\varepsilon_{ik} = 2e_{ik} + (e_{si} + \varphi_{si})(e_{sk} + \varphi_{sk}) = \varepsilon_{ki}. \tag{1,5}$$

In a body of homogeneous deformation, at each instant there are 3 rectilinear mutually perpendicular fibers (the principal axes of deformation[1]) along which the relative elongations are extremal. The limiting case of pure rotation without deformation is determined by the condition

$$\varepsilon_{ik} = 0 \quad \text{for all times } t. \tag{1,6}$$

The constraints (1,6) express the condition for the motion of the body as an absolutely rigid body.

Let us formulate the equations of motion, i.e., construct a mechanical theory (model) of a rotating and deformable body as a generalization of the model of an absolutely rigid body. To this end we use the principle of stationary action

$$\int_{t_0}^{t_1} \delta L\, dt = -\int_{t_0}^{t_1} \delta A\, dt. \tag{1,7}$$

The Lagrange function $L = T - U - U^{\text{ex}}$; U is the internal and U^{ex} the external potential energy; δA is the elementary virtual work of nonpotential forces. We formulate the equations of motion by choosing as generalized coordinates the 9 independent quantities

$$\varkappa_{ik} = u_{ik} + \delta_{ik}. \tag{1,8}$$

Let us compute the kinetic energy:

$$T = \frac{1}{2}\int v^2\, dm = \frac{1}{2}\int v^2 \rho\, dx_1\, dx_2\, dx_3; \tag{1,9}$$

ρ is the density. We obtain the velocity of the points by differentiating (1,2):

$$v_i = \dot{x}_i = \dot{\varkappa}_{ik} a_k. \tag{1,10}$$

Substituting (1,10) into (1,9) and changing to integration with respect to the variables a_1, a_2, a_3, we obtain

$$T = \frac{1}{2\rho_0} D\rho\, j^0_{sk} \dot{\varkappa}_{is} \dot{\varkappa}_{ik}; \tag{1,11}$$

D is the Jacobian of the transformation from x_i to a_i; j^0_{sk} are constants:

$$j^0_{sk} \equiv \rho_0 \int a_s a_k \, da_1 \, da_2 \, da_3 = j^0_{ks}. \tag{1,12}$$

From conservation of mass it follows that $m = \rho\tau = \rho_0 \tau_0$; τ is the volume of the body: $\tau = D\tau_0$, and consequently $\rho D = \rho_0$; therefore the kinetic energy takes the form

$$T = \frac{1}{2} j^0_{sk} \dot{\varkappa}_{is} \dot{\varkappa}_{ik} = \frac{1}{2} \left(\frac{1}{2} I_0 \dot{\varkappa}_{ik} \dot{\varkappa}_{ik} - I^0_{sk} \dot{\varkappa}_{is} \dot{\varkappa}_{ik} \right). \tag{1,13}$$

I^0_{sk} are the components of the inertia tensor—constants corresponding to the undeformed state,

$$I^0_{sk} \equiv j^0_0 \delta_{sk} - j^0_{sk}; \qquad j^0_0 = \frac{1}{2} I_0 \equiv j^0_{11} + j^0_{22} + j^0_{33}. \tag{1,14}$$

We represent the work of nonpotential forces in the form

$$\delta A = Q_{ik} \delta \varkappa_{ik}. \tag{1,15}$$

The Lagrange–Euler equations of the variational problem (1,7) have the usual form

$$\frac{d}{dt} p_{ik} \equiv j^0_{ks} \ddot{\varkappa}_{is} = -\frac{\partial}{\partial \varkappa_{ik}} \left(U + U^{\mathrm{ex}} \right) + Q_{ik}. \tag{1,16}$$

Here p_{ik} are the momenta conjugate to the coordinates $\varkappa_{ik}$:

$$p_{ik} \equiv \partial T / \partial \dot{\varkappa}_{ik} = j^0_{ks} \dot{\varkappa}_{is}. \tag{1,17}$$

The internal potential energy U should be regarded as a function of the components of the proper deformation ε_{ik}. Taking, for example, the model of an isotropic perfectly elastic body, we obtain

$$U = \tau\left(\frac{1}{2}\lambda\varepsilon^2 + \mu\varepsilon_{ik}\varepsilon_{ik}\right); \qquad \varepsilon \equiv \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}, \tag{1,18}$$

where the volume of the body is equal to

$$\tau = \tau_0\left(1 + \varepsilon + \varepsilon_{ik}\varepsilon_{ik} + \Delta\right), \tag{1,19}$$

where Δ is the determinant formed from the components ε_{ik}. For a liquid $\mu = 0$. For a general

in the case of an anisotropic body

$$U = \frac{1}{2\tau}\left(\lambda_{ikmn}\varepsilon_{ik}\varepsilon_{mn} + ...\right); \tag{1,20}$$

λ_{ikmn} is the tensor of elastic moduli of rank 4 (it has 21 independent components).

If the surface energy is comparable in order of magnitude with the volume energy, then to (1,18) one must also add the surface potential energy U^{surf}. For an isotropic film $U^{\mathrm{surf}} = aS$; a is the surface tension; S is the surface area, a definite function of the coordinates $\varkappa_{ik}$. We describe the viscous properties by means of the dissipation function Φ, which we regard as a function of the deformation velocity. For isotropic viscosity

$$\Phi = \frac{1}{2}\zeta\dot{\varepsilon}^2 + \eta\dot{\varepsilon}_{ik}\dot{\varepsilon}_{ik}. \tag{1,21}$$

The nonpotential forces have the form

$$Q_{ik} = -\partial\Phi/\partial\dot{\varkappa}_{ik} + Q'_{ik}, \tag{1,22}$$

where Q'_{ik} are still other possible nonpotential forces.

After specifying explicitly the form of U, Φ, and also U^{ex}, Q'_{ik} as functions of the generalized coordinates and velocities $\varkappa_{ik}$, $\dot{\varkappa}_{ik}$, we obtain a closed theory of a rotating and deformable body. The mechanical properties of the system (model) are described by 6 inertial constants I^0_{ks} (the mass m itself does not appear when the center of mass is fixed), by elastic constants λ, μ, and by viscous constants ζ, η. The theory is closed in the same sense as the theory of an absolutely rigid body, and is its generalization. To the three rotational degrees of freedom of an absolutely rigid body are added 6 deformation degrees of freedom ε_{ik}.

§ 2. There occur the rotation of the body, its deformation, and the interaction of rotation with deformation. To investigate these phenomena we transform the expression for the kinetic energy. We pass to Euler' s method of description; for this we solve the system (1,2):

$$a_i = v_{ik} x_k. \tag{2,1}$$

The coefficients v_{ik} are functions only of time; they form a matrix inverse to the matrix $\varkappa_{ik}$. Substituting (2,1) into (1,10), we obtain the velocity

$$v_i = \dot{\varkappa}_{ik} v_{ks} x_s \equiv V_i(x_1, x_2, x_3; t) = V_{ik} x_k + [\vec{\Omega}, \mathbf{r}]_i, \tag{2,2}$$

where

$$V_{ik} \equiv \frac{1}{2} \left(\partial V_i / \partial x_k + \partial V_k / \partial x_i \right) = V_{ki}(t); \tag{2,3}$$

$$\Omega_{ik} \equiv \frac{1}{2} \left(\partial V_i / \partial x_k - \partial V_k / \partial x_i \right) = -\Omega_{ki}(t);$$

$$\Omega_1 = -\Omega_{23}; \quad \Omega_2 = -\Omega_{31}; \quad \Omega_3 - \Omega_{12}. \tag{2,4}$$

The explicit expressions for V_{ik} and Ω_{ik} through $\dot{\varkappa}_{ik}$ have the form

$$V_{ik} = \frac{1}{2} \left(\dot{\varkappa}_{is} v_{sk} + \dot{\varkappa}_{ks} v_{si} \right); \quad \Omega_{ik} = \frac{1}{2} \left(\dot{\varkappa}_{is} v_{sk} - \dot{\varkappa}_{ks} v_{si} \right). \tag{2,5}$$

Substituting (2,2) into (1,9), we obtain the kinetic energy in the form

$$T = \frac{1}{2} j_{ik} V_{in} V_{kn} + \frac{1}{2} I_{ik} \Omega_i \Omega_k + \Omega_{iL} i. \tag{2,6}$$

The moments of inertia I_{ik} are functions of time through the coordinates $\varkappa_{ik}$, namely:

$$I_{ik} \equiv j \delta_{ik} - j_{ik}; \quad j \equiv j_{11} + j_{22} + j_{33} \equiv \frac{1}{2} I; \quad j_{ik} = j^0_{sn} \varkappa_{is} \varkappa_{kn}. \tag{2,7}$$

The axial vector L_i is defined by the equalities:

$$L_1 \equiv -L_{23}, \ldots; \quad L_{ik} \equiv I_{in} V_{kn} - I_{kn} V_{in} \equiv -L_{ki}. \tag{2,8}$$

The kinetic energy (2,6) now has a clear interpretation. The term

$$T_{\text{df}} = \frac{1}{2} j_{ik} V_{in} V_{kn} \tag{2,9}$$

describes the kinetic energy of "instantaneous deformation" ; the terms

$$T_{\text{rot}} = \frac{1}{2} I_{ik} \Omega_i \Omega_k; \qquad T_{\text{rd}} = \Omega_i L_i \tag{2,10}$$

are the kinetic energies of "instantaneous rotation" and of the "instantaneous interaction of rotation with deformation." The quantities V_{ik} and Ω_{ik} are velocities of nonholonomic coordinates; therefore the equations of motion for them have a cumbersome form. However, T_{df} is easily expressed directly in terms of the deformation velocities $\dot{\varepsilon}_{ik}$. It is not difficult to obtain the relation

$$\dot{\varepsilon}_{ik} = (\varkappa_{ni} \varkappa_{sk} + \varkappa_{si} \varkappa_{nk}) V_{sn}. \tag{2,11}$$

Denote the solution of the system (2,11) with respect to V_{sn} by

$$V_{nm} = b_{nm,ik} \dot{\varepsilon}_{ik}. \tag{2,12}$$

Substituting (2,12) into (2,6), we obtain the kinetic energy in the form

$$T = \frac{1}{2} J_{nm,rs} \dot{\varepsilon}_{nm} \dot{\varepsilon}_{rs} + \frac{1}{2} I_{ik} \Omega_i \Omega_k + \Omega_{iL} i, \tag{2,13}$$

$$J_{nm,rs} = j_{ik}\, b_{ip,nm} b_{kp,rs} = J_{rs,nm}.$$

Now one may choose as generalized coordinates the 6 ε_{ik} and also 3 quantities independent of them, for example φ_{ik} from (1,3). The equations of motion for these coordinates are equations of the usual Lagrange–Euler form.

§ 3. Let us form the momenta conjugate to the nonholonomic angular coordinates $d\chi_i \equiv \Omega_i dt$. From (2,6) and (2,13) we obtain

$$M_i \equiv (\partial T / \partial \Omega_i)_{V_{nm}} = (\partial T / \partial \Omega_i)_{\dot{\varepsilon}_{nm}} = I_{ik} \Omega_k + L_i. \tag{3,1}$$

It is easy to prove that $\mathbf{M}$ has the meaning of the total angular momentum, determined by the equality

$$\mathbf{M} = \int [\mathbf{r}, \mathbf{V}]\, dm = M_i \mathbf{e}_i.$$

From (3,1) the meaning of the axial vector L_i is clear: it is the angular momentum of the deformation of the body itself. Let us find the law of variation of M_i in time; for this purpose we express M_i through $\varkappa_{ik}$. Simple calculations give

$$M_{ik} = \varkappa_{ks} p_{is} - \varkappa_{is} p_{ks} = -M_{ki}; \quad M_1 = -M_{23}, \dots \tag{3,2}$$

Multiplying the equations of motion (1,16) by $\varkappa_{rk}$, after some transformations we obtain:

$$dM_{ik}/dt = \varkappa_{ks} Q_{is} - \varkappa_{is} Q_{ks} - (\varkappa_{ks} \partial U^{\text{ex}} / \partial \varkappa_{is} - \varkappa_{is} \partial U^{\text{ex}} / \partial \varkappa_{ks}); \tag{3,3}$$

here it has been taken into account that

$$\varkappa_{ks} \partial U / \partial \varkappa_{is} - \varkappa_{is} \partial U / \partial \varkappa_{ks} = 0. \tag{3,4}$$

(3,4) follows from the fact that U is a function only of the deformation coordinates ε_{rs}. On the right-hand side of (3,3) stands an axial vector having the meaning of the total moment of all external forces applied to the body, so that in vector form this equation has the usual form of the law of variation of the total angular momentum of a system of points. Three equations (3,3) are equivalent to 3 Lagrange equations for φ_{ik}, obtained using (2,13), or to any other 3 equations corresponding to 3 orientational coordinates ϑ_r which in some way determine the rotation of the body.

The closed theory set forth can be applied to the study of various problems of classical mechanics, for example the rotation of a deformable top.

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References

1. V. V. Novozhilov, *Foundations of the Nonlinear Theory of Elasticity*, 1948.

Note: Figure translations are in progress. See original paper for figures.