# ON SOME PROPERTIES OF SUMMATION METHODS

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## Abstract

## Full Text

## MATHEMATICS

**S. P. GEISBERG**

# ON SOME PROPERTIES OF SUMMATION METHODS

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**Notation.** $A = (a_{nk})$ is a complex matrix; $A^{\varphi} = (a_{n\varphi_k})$ is a submatrix of the matrix $A$, in which the second index runs through the values of a subsequence $\varphi = \{\varphi_k\}$ of the natural numbers. If $A$ is the matrix $(a_{nk})$, then by $A_1, A_2, A_3$ we denote the sets of sequences $\xi = \{\xi_k\}$ of complex numbers for which, respectively*:

$$\sum_n \left| \sum_k a_{nk}\xi_k \right| < \infty;$$

$$\sup_n \left| \sum_k a_{nk}\xi_k \right| < \infty;$$

the sequence $\eta = \{\eta_n\}$, $\eta_n = \sum_k a_{nk}\xi_k$, converges.

Further, $Q = \{q_k(x)\}$ is a sequence of functions continuous on $[0, \infty)$; $Q^{\varphi} = \{q_{\varphi_k}(x)\}$ is its subsequence; $Q_1, Q_2, Q_3$ are the sets of sequences $\xi = \{\xi_k\}$ for which, respectively,

$$\int_0^{\infty} \left|\sum q_k(x)\xi_k\right| dx < \infty;$$

$$\sup_{0<\alpha<\beta<\infty} \left| \int_{\alpha}^{\beta} \left(\sum q_k(x)\xi_k\right) dx \right| < \infty;$$

there exists

$$\int_0^{\infty} \left(\sum q_k(x)\xi_k\right) dx.$$

The matrix $A$ (the sequence $Q$) is called a summation method; the sets $A_3, A_2, A_1$ $(Q_3, Q_2, Q_1)$ are called, respectively, the fields of summability, boundedness, and absolute summability.

By $c, c_0, m$ we denote, respectively, the sets of convergent sequences, sequences converging to zero, and bounded sequences; $e$ is the set of sequences $\xi = \{\xi_k\}$ for which $\sum |\xi_k| < \infty$.

For a matrix $A$ and a sequence $\xi = \{\xi_k\}$ we put

$$\|\xi\|^p_{A_1} = \sum_n \left|\sum_{k=p}^{\infty} a_{nk}\xi_k\right|; \qquad \|\xi\|^p_{A_2} = \|\xi\|^p_{A_3} = \sup_n \left|\sum_{k=p}^{\infty} a_{nk}\xi_k\right|;$$

$$\|\xi\|^p_{A_1} = \sum_{k=p}^{\infty} |\xi_k|; \qquad \|\xi\|^p_2 = \|\xi\|^p_3 = \sup_{k \geq p} |\xi_k|.$$

Similarly, for sequences $Q$ and $\xi = \{\xi_k\}$ we put

$$\|\xi\|^p_{Q_1} = \int_0^{\infty} \left|\sum q_k(x)\xi_k\right| dx; \qquad \|\xi\|^p_{Q_2} = \|\xi\|^p_{Q_3} = \sup_{0<\alpha<\beta<\infty} \left|\int_{\alpha}^{\beta} \left(\sum q_k(x)\xi_k\right) dx\right|.$$

In § 1 of the present note we obtain some relations between concrete summation methods; § 2 is devoted to the study of the interrelations of the fields of different summation methods. In § 3 we give several properties of operators in the space $c_0$, analogous to properties of matrix summation methods.

* If the limits of summation are not indicated, summation is carried out over all integer values of the index fr

§ 1. We shall consider the following methods:

1. The Cesàro method of order $a$, or the method $(C, a)$ $(a > 0)$:

$$a_{nk} = \begin{cases} 1, & n = k = 0, \\ \dfrac{k\binom{n-k+a-1}{a-1}}{n\binom{n+a}{a}}, & n \geqslant 1;\ n \geqslant k \geqslant 0, \\ 0, & k > n. \end{cases}$$

2. The Euler method of order $a$, or the method $(E, a)$ $(a > 0)$:

$$a_{nk} = \begin{cases} \dfrac{a^{n-k}}{(a+1)^{n+1}}\binom{n}{k}, & n \geqslant k \geqslant 0, \\ 0, & k > 0. \end{cases}$$

3. Abel’ s method, or method $A$:

$$q_k(x) = ke^{-kx}.$$

4. The integral Borel method, or method $B$:

$$q_k(x) = e^{-x}\frac{x^k}{k!}.$$

**Theorem 1.** In order that: a) $(C,a)_1^\varphi \subset e$; b) $A_1^\varphi \subset e$, it is necessary and sufficient that there exist an index $s$ and a number $r > 1$ such that, for $k > s$,

$$\varphi_{k+1}/\varphi_k > r. \tag{1}$$

**Theorem 2.** In order that $(E,a)_1^\varphi \subset e$, it is necessary and sufficient that the following conditions be fulfilled:

1) If $\xi = \{\xi_k\} \in (E,a)_1^\varphi$, then $\xi_k = O(\gamma^k)$ for some $\gamma > 0$.
2) There exist an index $p$ and a number $q > 0$ such that, for all $k > p$,

$$\varphi_{k+1} - \varphi_k > q\sqrt{\varphi_k}. \tag{2}$$

**Theorem 3.** In order that $B_1^\varphi \subset e$, it is necessary and sufficient that the following conditions be fulfilled:

1) If $\xi = \{\xi_k\} \in B_1^\varphi$, then $\xi_k = O(\gamma^k)$ for some $\gamma > 0$.
2) There exist an index $p$ and a number $q > 0$ such that, for all $k > p$,

$$\varphi_{k+1} - \varphi_k > q\sqrt{\varphi_k}. \tag{3}$$

The sufficiency of condition (1) in case a) of Theorem 1 was proved in paper (1). The necessity of conditions (1), (2), (3) is proved with the aid of the following two theorems, which are of independent interest.

**Theorem 4.** Let the method $A$ be given by the matrix $(a_{nk})$, and suppose that: 1) $e \in A_1$; 2) $(m-e) \cap A_1 = \emptyset$; 3) $\sum_k |a_{nk}| < \infty$ $(n = 0, 1, \ldots)$. Then there exist an index $p$ and a number $q > 0$ such that, for all $\xi \in e$, we have

$$\|\xi\|_{A_1}^p > q\|\xi\|_1^p.$$

**Theorem 5.** Let the method $Q$ be given by a sequence $\{q_k(x)\}$ of functions continuous on $[0,\infty)$, and suppose that: 1) $e \in Q_1$; 2) $(m-e) \cap Q_1 = \emptyset$; 3) $\sum_k |q_k(x)| < \infty$ $(x \in [0,\infty))$. Then there exist an index $p$ and a number $q > 0$ such that, for all $\xi \in e$, we have

$$\|\xi\|_{Q_1}^p > q\|\xi\|_1^p.$$

Theorems analogous to Theorems 1 and 2 for summability fields are proved in (2); however, the necessity of conditions (1), (2), (3) is established there only for condition (1) in the case of the method $(C, \alpha)$, $0 < \alpha \leqslant 1$. For the remaining cases, necessity can be established by the same device as in the proof of Theorems 1 and 2, with the aid of Theorem 9 established below.

§ 2. Let $A$ and $B$ be two summability methods given by matrices $(a_{nk})$ and $(b_{nk})$, respectively, with

$$\sum_k (|a_{nk}| + |b_{nk}|) < \infty, \qquad n = 0, 1, \dots .$$

**Theorem 6.** If for some $i$ and $j$, $1 \leqslant i, j \leqslant 3$, we have $A_i \subset B_j$, then for every $\delta > 0$ there exist an index $p(\delta)$ and a number $q(\delta) > 0$ such that for all $\xi \in A_i \cap m$ we have

$$\|\xi\|_{A_i}^{(\delta)} + \delta q(\delta)\|\xi\|_2^{p(\delta)} > q(\delta)\|\xi\|_{B_j}^{p(\delta)}.$$

**Remark.** For finite-row matrices one may put $\delta = 0$ $(p(0) < \infty,\ q(0) > 0)$.

For methods $P$ and $Q$, given by the sequences $\{p_k(x)\}$ and $\{q_k(x)\}$, with

$$\sum_k (|p_k(x)| + |q_k(x)|) < \infty,$$

analogous relations hold if everywhere $A_i$ is replaced by $P_i$ and $B_i$ by $Q_i$.

As consequences of Theorem 6 we obtain:

**Theorem 7.** If $A_1 \subset e$, then there exist an index $p$ and a number $q > 0$ such that, for all $\xi \in A_1 \cap m$,

$$\|\xi\|_{A_1}^p > q\|\xi\|_1^p.$$

**Theorem 8.** If $A_1 \subset m$, then there exist an index $p$ and a number $q > 0$ such that, for all $\xi \in A_1 \cap m$,

$$\|\xi\|_{A_1}^p > q\|\xi\|_2^p.$$

**Theorem 9.** If $A_2 \subset m$ $(A_3 \subset m)$, then there exist an index $p$ and a number $q > 0$ such that for all $\xi \in A_2 \cap m$ $(\xi \in A_3 \cap m)$ we have

$$\|\xi\|_{A_2}^p > q\|\xi\|_2^p.$$

Theorem 9 for the case $A_3 \subset m$, with the matrix $A$ the matrix of Toeplitz, is asserted in ($^3$). With the help of Theorem 9 the following result, due to Zeller ($^4$), is easily proved:

**Theorem 10.** If $c \subset A_3 \subset m$, then $c = A_3$.

Theorem 8 leads to an analogue of Theorem 10 for fields of absolute summability.

**Theorem 11.** If $e \subset A_1 \subset m$ and, for every $p$, the bounded solutions of the system of equations

$$\sum_n \tau_n a_{nk} = 0 \qquad (k = p,\ p+1, \ldots)$$

are contained in $e$, then $A_1 \subset c$.

Applying Theorem 8 to methods 1 and 2 and using the results of § 1, we obtain:

**Theorem 12.** If $(C, \alpha)_1^\varphi \subset m$, then $(C, \alpha)_1^\varphi = e$.

**Theorem 13.** If $(E, \alpha)_1^\varphi \subset m$, then $(E, \alpha)_1^\varphi = e$.

§ 3. Introduce on $c_0$ the norm $\|\xi\| = \sup_k |\xi_k|$. Then $c_0$ will be a $B$-space. Let $A$ be a linear continuous operator mapping $c_0$ into itself. Denote by $c_n$ the $n$-th adjoint $c_0$-space and by $A_n$ the $n$-th adjoint operator to $A$.

**Theorem 14.** If from $(A_4, x) \in c_0$ it follows that $x \in c_2$, then from $(A_4, x) \in c_0$ it follows that $x \in c_0$; moreover, there exists $q > 0$ such that for all $x \in c_0$

$$\|(A_4, x)\| > q\|x\|.$$

Further, it is not difficult to see that the operator $A_1$ can be represented in the form

$$\eta = (A_1, \xi), \qquad \eta = \{\eta_n\}, \qquad \xi = \{\xi_k\}, \qquad \eta_n = \sum_k a_{nk}\xi_k,$$

where the matrix $(a_{nk})$ depends only on the operator $A$.

**Theorem 15.** If from $(A_4, x) \in c_1$ it follows that $x \in c_2$, and for each $p$ the bounded solutions of the system of equations

$$\sum_n \tau_n a_{nk} = 0 \qquad (k = p,\ p+1, \ldots)$$

are contained in $c_1$, then from $(A_4, x) \in c_1$ it follows that $x \in c_0$.

Theorems 14 and 15 are, obviously, in a certain sense analogues of Theorems 10 and 11.

Tartu State University

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*Note: Figure translations are in progress. See original paper for figures.*