A. V. STRAUS

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Abstract

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A. V. STRAUS

ON SOME FAMILIES OF EXTENSIONS OF A SYMMETRIC OPERATOR

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In the present note we study some questions in the theory of extensions of symmetric operators that are adjacent to spectral theory and are connected with results of M. A. Naimark ((*1*), see also (*2*)). On this basis we consider a singular boundary-value problem for the equation $l[y]-\lambda y=0$ with boundary conditions depending analytically on λ, where $l[y]$ is a formally self-adjoint ordinary differential expression of even order. Boundary-value problems of this type arise, in particular, from the application of the Fourier method in the study of oscillations of a part of a conservative system. The appearance of the parameter λ in the boundary conditions is due to the fact that these conditions may depend on such, for example, frequency characteristics of the remaining part of the system as its dynamical compliance.

1. Let A be a closed symmetric operator in the Hilbert space H with dense domain of definition D_A. Consider the equation

$$\widetilde{A}\tilde{f}-\lambda\tilde{f}=0, \tag{1}$$

where $\widetilde{A}$ is some self-adjoint operator in a Hilbert space $\widetilde{H}\supset H$, which is an extension of the operator A. Suppose that it is required to find not the eigenvectors themselves of the operator $\widetilde{A}$, but only their projections onto H. Let P and $\widehat{P}$ be the operators of orthogonal projection in $\widetilde{H}$ onto the subspaces H and $\widetilde{H}\ominus H$, respectively. Denote by $\mathfrak{B}_\lambda$, for each complex λ, the manifold of all $\tilde{f}\in D_{\widetilde{A}}$ for which $\widehat{P}(\widetilde{A}-\lambda E)\tilde{f}=0$. Define an operator B_λ in H by putting $D_{B_\lambda}=P\mathfrak{B}_\lambda$, $B_\lambda P\tilde{f}=P\widetilde{A}\tilde{f}$ $(\tilde{f}\in\mathfrak{B}_\lambda)$. This definition is correct, and for every complex λ we have $A\subset B_\lambda\subset A^*$.* We shall agree to say that the family of operators B_λ, depending on the complex parameter λ, thus defined is induced in H by the self-adjoint extension $\widetilde{A}$ of the operator A.

The following proposition is valid: for every complex λ the set of all solutions f of the equation

$$B_\lambda^* f-\lambda f=0 \tag{2}$$

coincides with the set of all possible elements $P\tilde{f}$, where $\tilde{f}$ satisfies equation (1) with the same λ.*

Thus, if we are interested only in the projections of the eigenvectors of the operator $\widetilde{A}$ onto H, then equation (1) may be replaced by equation (2). Therefore it is not necessary to assume the operator $\widetilde{A}$ known; it is sufficient to specify the family B_λ induced by it in H. The question naturally arises of the intr–

* It follows from this that for non-real λ equation (2) has only the zero solution.

characteristic of the families B_λ, which are induced in H by different self-adjoint extensions of the given symmetric operator A.

For any real λ the operator B_λ is symmetric. For any non-real λ the operator $B_\lambda - \lambda E$ has a bounded inverse defined on all of H, and for any $h \in H$

$$(B_\lambda - \lambda E)^{-1} h = P(\widetilde{A} - \lambda E)^{-1} h \qquad (\operatorname{Im} \lambda \neq 0). \tag{3}$$

According to formula (3), the operator $(B_\lambda - \lambda E)^{-1}$, for any non-real λ, coincides with the generalized resolvent R_λ of the operator A generated by the self-adjoint extension $\widetilde{A}$ (see, for example, ([2]), p. 377).

Denote by $\mathfrak{N}_\mu$, for any non-real μ, the defect subspace of the operator A, consisting of all solutions of the equation $A^*\varphi - \mu\varphi = 0$. If F is some linear operator with domain $D_F = \mathfrak{N}_\mu$ and with range in $\mathfrak{N}_{\bar{\mu}}$, then by A_F we denote the operator in H with domain

$$D_{A_F} = D_A + (F - E)\mathfrak{N}_\mu,$$

satisfying the relation

$$A \subset A_F \subset A^*.$$

The formula obtained by the author in ([3]), Theorem 7, for all generalized resolvents of a symmetric operator makes it possible to describe in the following way all possible families B_λ for non-real λ.

Theorem 1. *A family of operators B_λ, depending on a non-real parameter λ, is induced in H by some self-adjoint extension of the operator A if and only if B_λ admits the representation*

$$B_\lambda = \begin{cases} A_{F(\lambda)} & \text{for } \operatorname{Im}\lambda \cdot \operatorname{Im}\lambda_0 > 0, \\ A_{F^*(\bar{\lambda})} & \text{for } \operatorname{Im}\lambda \cdot \operatorname{Im}\lambda_0 < 0, \end{cases} \tag{4}$$

where λ_0 is a fixed non-real number; $F(\lambda)$ is a certain linear operator from $\mathfrak{N}_{\lambda_0}$ into $\mathfrak{N}_{\bar{\lambda}_0}$, depending on λ and satisfying the following conditions: 1) $\|F(\lambda)\| \leq 1$

$(\operatorname{Im}\lambda \cdot \operatorname{Im}\lambda_0 > 0)$*; 2)* $F(\lambda)$ *is an analytic operator function of* λ *in the half-plane* $(\operatorname{Im}\lambda \cdot \operatorname{Im}\lambda_0 > 0)$.

This result does not give a description of the family B_λ for real values of λ, which are of special interest, since only for such λ can equation (2) have nonzero solutions.

Taking formula (3) into account, it is not difficult to prove that the operator function B_λ of the complex parameter λ is continuous at any non-real point μ in the following sense: as $\lambda \to \mu$, the gap between the graphs of the operators B_λ and B_μ tends to zero*. At points of the real axis the operator function B_λ is not always continuous. The following theorem gives a complete description of those operator functions B_λ that turn out to be continuous in the whole complex plane.

Theorem 2. *Let* A *be a closed symmetric operator in* H *with equal defect numbers. A family of operators* B_λ *is induced in* H *by some self-adjoint extension of the operator* A*, and* B_λ *is a continuous operator function of the parameter* λ *in the whole complex plane if and only if the operator function* $F(\lambda)$ *occurring in formula (4), in addition to conditions 1) and 2) of Theorem 1, also satisfies the following: 3)* $F(\lambda)$ *is an operator function of the parameter* λ*, continuous in the sense of uniform convergence of operators, in the closed half-plane* $(\operatorname{Im}\lambda \cdot \operatorname{Im}\lambda_0 \geq 0)$*; 4) for any real* λ*,* $F(\lambda)$ *is an isometric operator mapping* $\mathfrak{N}_{\lambda_0}$ *onto* $\mathfrak{N}_{\bar\lambda_0}$*. If all these conditions are fulfilled, formula (4) remains valid also for any real* λ*, i.e.*

$$B_\lambda = A_{F(\lambda)} = A_{F^*(\lambda)} \qquad (\operatorname{Im}\lambda = 0).$$

* The definition of the gap between two linear manifolds is given, for example, in (2), pp. 105-108.

2. Let $l[y]$ be a formally self-adjoint ordinary differential expression of order $2n$ with real coefficients satisfying, on the interval (a, b), the conditions of local summability. The expression $l[y]$ generates in the space $\mathcal{L}^2(a, b)$ a closed symmetric differential operator $\mathcal{L}$ with minimal domain of definition (2). Suppose that the endpoint a is regular, b is singular, and that the operator $\mathcal{L}$ has deficiency index (n, n). If $y(x)$ is a function for which the expression $l[y]$ has meaning, then by $\hat{y}(x)$ we denote the vector-valued function

$$(y(x), y^{[1]}(x), \dots, y^{[2n-1]}(x)),$$

where $y^{[k]}(x)$ is the quasi-derivative of order k, and we shall regard $\hat{y}(x)$ as a column matrix. Define the square matrix

$$J = \|\varepsilon_{ik}\|_1^{2n}$$

by the equalities: $\varepsilon_{ik} = 0$ for $i + k \neq 2n + 1$; $\varepsilon_{ik} = \operatorname{sgn}(i - k)$ for $i + k = 2n + 1$ $(i, k = 1, 2, \dots, 2n)$.

Introduce for consideration the class K of all possible matrix functions $U(\lambda)$ of the complex parameter λ, composed of n rows and $2n$ columns and satisfying the following conditions: a) $U(\lambda)$ is a meromorphic function of λ, having no singularities on the real axis; b) for every non-real λ distinct from the poles of the function $U(\lambda)$,

$$\frac{1}{\lambda-\bar{\lambda}}\,U(\lambda)JU^*(\lambda)\leq 0;$$

c) the rank of the matrix $U(\lambda)$, for every λ distinct from its poles, is equal to n.

With the aid of Theorem 2 the following proposition is proved:

Theorem 3. *Let $U(\lambda)$ be an arbitrary matrix function from the class K, and for every λ distinct from its poles let the operator B_λ in $\mathcal{L}^2(a,b)$ be defined as a part of the operator $\mathcal{L}^*$, given on the manifold D_{B_λ} of all*

$$y(x)\in D_{\mathcal{L}^*},$$

which satisfy the boundary condition

$$U(\lambda)\hat{y}(a)=0.$$

Then the family B_λ is induced in $\mathcal{L}^2(a,b)$ by some self-adjoint extension of the operator $\mathcal{L}$, and the operator-valued function B_λ, extended in the corresponding way at the poles of the function $U(\lambda)$, is continuous in the whole complex plane. Conversely, if a family of operators B_λ is induced in $\mathcal{L}^2(a,b)$ by some self-adjoint extension of the operator $\mathcal{L}$, and the operator-valued function B_λ is continuous in the whole complex plane, then in the class K there exists a matrix function $U(\lambda)$ by means of which the family B_λ is defined in the manner indicated above.

Consider the boundary-value problem:

$$l[y]-\lambda y=0; \tag{5}$$

$$U(\lambda)\hat{y}(a)=0; \tag{6}$$

$$y(x)\in\mathcal{L}^2(a,b), \tag{7}$$

where $U(\lambda)$ is some matrix function from the class K.* This problem may be written in the form of the equation

$$B_\lambda y-\lambda y=0,$$

where B_λ is the family of extensions of the operator $\mathcal{L}$ described in Theorem 3.

Let

$$z_1(x;\lambda),\ z_2(x;\lambda),\dots,\ z_n(x;\lambda) \tag{8}$$

be linearly independent solutions of equation (5), satisfying the boundary condition (6), defined for all real values of the parameter λ and continuous with respect to λ on the entire real axis. The column matrix composed of the functions (8) will be denoted by $z(x;\lambda)$, and by $z'(x;s)$ the transposed row matrix.

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* An analogous boundary-value problem for an equation of second order was studied in ([4]).

Theorem 4. For any function $f(x) \in \mathcal{L}^2(a,b)$ the expansion

$$f(x) = \int_{-\infty}^{+\infty} z'(x;\sigma)\, dS(\sigma)\, \xi(f;\sigma), \tag{9}$$

holds, where

$$\xi(f;\sigma) = \int_a^b f(S)\, \overline{z(S;\sigma)}\, dS, \tag{10}$$

and $S(\sigma)$ is some Hermitian matrix of order n, a nondecreasing function of the parameter σ $(-\infty < \sigma < +\infty)$; the integrals in formulas (9) and (10) converge respectively in the metrics of $\mathcal{L}^2(a,b)$ and $\mathcal{L}^2_S(-\infty,+\infty)$*. Moreover, the equality

$$\int_a^b |f(x)|^2\, dx = \int_{-\infty}^{+\infty} \xi^*(f;\sigma)\, dS(\sigma)\, \xi(f;\sigma)$$

is valid.

Because of lack of space we do not give here the formulas yielding an effective rule for constructing the matrix function $S(\sigma)$. Let us note that the eigenvalues of the boundary-value problem (5)–(7) coincide with the points of discontinuity of the function $S(\sigma)$, and the eigensubspace corresponding to the eigenvalue $\lambda = \sigma$ has dimension equal to the rank of the matrix $S(\sigma+0) - S(\sigma-0)$.

Formula (9) is a realization of the abstract expansion

$$f = \int_{-\infty}^{+\infty} dE_t f \quad (f \in H),$$

where E_t $(-\infty < t < +\infty)$ is the spectral function of the symmetric operator A in H, connected with the spectral function $\widetilde{E}_t$ $(-\infty < t < +\infty)$ of the self-adjoint operator $\widetilde{A} \supset A$ in $\widetilde{H} \supset H$ by M. A. Naimark's formula (see, for example, ([2]), pp. 370–372)

$$E_t f = P\widetilde{E}_t f \quad (f \in H).$$

In the case considered here $H = \mathcal{L}^2(a,b)$, $A = \mathcal{L}$, and the operator $\widetilde{A}$, which is not assumed to be known, induces in $\mathcal{L}^2(a,b)$ a family of operators $B_\lambda \supset \mathcal{L}$, defined by means of the boundary conditions (6).

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3. A. V. Shtraus, *Izv. AN SSSR, ser. matem.*, **18**, No. 1, 51 (1954).

4. A. V. Shtraus, *Izv. AN SSSR, ser. matem.*, **20**, No. 6, 783 (1956).

* $\mathcal{L}^2_S(-\infty, +\infty)$

is the Hilbert space of n-dimensional vector functions

$$\eta(\sigma) = (\eta_1(\sigma), \eta_2(\sigma), \dots, \eta_n(\sigma)) \quad (-\infty < \sigma < +\infty),$$

regarded as column matrices; the scalar product is defined by the formula

$$(\eta, \chi) = \int_{-\infty}^{+\infty} \chi^*(\sigma)\, dS(\sigma)\, \eta(\sigma).$$

Note: Figure translations are in progress. See original paper for figures.

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