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Abstract

Full Text

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ON SOME WELL-POSED PROBLEMS FOR ULTRAHYPERBOLIC AND WAVE EQUATIONS WITH DATA ON THE CHARACTERISTIC CONE

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Let us consider the following problem of integral geometry: the integrals of a certain function $G(z)/|z|^{n-2}$ over any sphere passing through the origin of coordinates are given; determine the function $G(z)$, i.e., solve the integral equation:

$$\psi(x) = \frac{1}{|x|} \int_{|z|^2=(x,z)} \frac{G(z)}{|z|^{n-2}} \, dS_z, \qquad x = x_1 \dots x_n; \; z = z_1 \dots z_n. \tag{1}$$

Let n be odd. Introduce the operator P, defined by the equality

$$P\varphi = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \frac{1}{|z|^{\frac{n-1}{2}}} \frac{\partial^{\frac{n-1}{2}}}{\partial |z|^{\frac{n-1}{2}}} \int_{(x,z)=|z|^2} \varphi(x) \, dS_x. \tag{2}$$

From the Plancherel formula for the Radon transform (2) it follows that the operator P is isometric in L_2. It turns out that in fact P is unitary. The proof can be carried out by constructing $S = P^*$ and establishing the unitary equivalence of the operators S and P.

It can be shown that the operator S has the form

$$S\psi = \frac{1}{(-2\pi)^{\frac{n-1}{2}}} \frac{\partial^{\frac{n-1}{2}}}{\partial |x|^{\frac{n-1}{2}}} \left[|x|^{\frac{n-3}{2}} \int_{(z,x)=|z|^2} \frac{\psi(z)}{|z|^{n-2}} \, dS_z \right]. \tag{3}$$

Further, it is easy to obtain that

$$KSK = P, \tag{4}$$

where the operator K assigns to the function $\varphi(x)$ the function

$$f(\xi) = K\varphi = \frac{\varphi(\xi/|\xi|^2)}{|\xi|^n}.$$

Since the operator K is unitary and self-adjoint, equality (4) establishes the required unitary equivalence of P and S.

We note the curious fact that the operator $KP = SK$ is unitary and self-adjoint, and thus L_2 decomposes into two orthogonal subspaces, on each of which the operator P coincides, up to sign, with K.

The results given above allow us to formulate the following theorem:

Theorem. Let $\psi(x)$ $(x = x_1 \dots x_n)$ be an arbitrary $n + 1$ times continuously differentiable function, defined for all x, and suppose that there exist–

there exists a constant $C > 0$ such that

$$\left| \frac{\partial^k \psi(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right| \leq \frac{C}{1 + |x|^{\frac{3n+1}{2}}} \quad \left(0 \leq \sum_{i=1}^{n} \alpha_i = k \leq n + 1 \right). \tag{5}$$

Then there exists a unique function $G(z)$ of L_2, continuous in the whole space and twice continuously differentiable everywhere except at the origin, satisfying equation (1). The function $G(z)$ is found by the formula

$$G(z) = \frac{(-1)^{\frac{n-1}{2}}}{(2\pi)^{n-1}} \frac{\partial^{n-1}}{\partial |z|^{n-1}} \int_{(x,z)=|z|^2} \psi(x) \, dS_x. \tag{6}$$

Completely analogous results hold in the even-dimensional case. The theorem formulated makes it possible to solve several problems for the ultrahyperbolic and wave equations.

Problem I. Find a function of $2n$ variables $U(x_1, \dots, x_n, y_1, \dots, y_n) \equiv U(x, y)$, satisfying, for $|x| < |y|$, the ultrahyperbolic equation

$$\Delta_x U = \Delta_y U \tag{7}$$

and the boundary condition

$$U\Big|_{|x|=|y|} = \varphi(x, y) * . \tag{8}$$

Let $\varphi(x, y)$ have the property that

$$\left| \frac{\partial^j \varphi(x, y)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n} \partial y_1^{\beta_1} \dots \partial y_n^{\beta_n}} \right| \leq \frac{C}{1 + |x|^{\frac{5n-1}{2}}} \quad \left(0 \leq \sum \alpha_i + \sum \beta_i = j \leq n + 1 \right). \tag{9}$$

Suppose that the solution $U(x, y)$ exists. Then Asgeirsson' s theorem ([1]) is applicable to it:

$$\int_{|z|^2=r^2} U(x+z,y)\,dS_z = \int_{|z|^2=r^2} U(x,y+z)\,dS_z.$$

Putting $x = 0,\ r = |y|$ in the last formula, we obtain:

$$\int_{|z|^2=|y|^2} U(0,y+z)\,dS_z = \int_{|z|^2=|y|^2} \varphi(z,y)\,dS_z. \tag{10}$$

But (10) is nothing other than an integral equation of the form (1) for the function $U(0,y)$. Solving it, we find the value of the function U at any point of the form $\{0,y\}$. To find U at an arbitrary point $\{x,y\}$ ($|x| < |y|$), we can, by means of ultrahyperbolic transformations, carry the point $\{x,y\}$ into a point of the form $\{0,y'\}$ and thus reduce the problem to the one already solved.

The solution obtained can be represented in the form

$$U(x,y) = \frac{C_n\big||y|^2-|x|^2\big|}{\Gamma\left(\frac{\lambda+1}{2}\right)} \int \frac{\varphi(\xi,\eta)}{|\xi|}\,\big||y-\eta|^2-|x-\xi|^2\big|^\lambda\, dS_{\xi,\eta}\Big|_{\lambda=-n}. \tag{11}$$

The integral in formula (11) is taken over the whole cone $|\xi|^2 = |\eta|^2$ and is understood in the sense of analytic continuation in λ from sufficiently large values of λ; C_n is a constant depending on the dimension of the space. The function $U(x,y)$ is unique in the class of functions satisfying the inequality

$$|U(x,y)| \le \frac{A(x)}{(|y|^2-|x|^2)^{\frac{n-2}{2}}\left[1+(|y|^2-|x|^2)^{\frac{n+1}{4}}\right]},$$

* We note that Problem I was considered by N. S. Piskunov ([3,4]). However, he restricted himself to the case $n = 2$ and assumed that the function $\varphi(x,y)$ was expandable in a Fourier series in the angular variables with Fourier coefficients that are analytic functions of $|x|$.

where $A(x)$ is a positive function bounded in bounded domains.

It can be shown that the function U constructed in this way is indeed a solution of Problem I.

It is also not hard to see that the solution (11) depends continuously on the function $\varphi(\xi,\eta)$: if $\varphi_m(\xi,\eta)$ satisfies the estimates (9) uniformly in m and $\varphi_m(\xi,\eta)$ converges to $\varphi_0(\xi,\eta)$ uniformly in (ξ,η), together with its derivatives up to order $n+1$ inclusive, then $U_m(x,y) \to U_0(x,y)$, together with the first and second derivatives, uniformly in any closed, bounded interior subdomain, and $U_0(x,y)$ is a solution of Problem I with boundary condition

$$U_0\big|_{|x|=|y|} = \varphi_0(x,y).$$

Problem II. Find the function $U(x,t)$ outside the characteristic cone $|x|^2 = t^2$, satisfying the wave equation

$$U_{tt} = \Delta_x U \quad \text{for } t < |x|,\ t > 0 \tag{12}$$

and the boundary conditions

$$U\big|_{t=|x|} = \varphi(x); \tag{13}$$

$$U\big|_{t=0} = 0. \tag{14}$$

With respect to $\varphi(x)$ we assume that it satisfies the conditions (5). Suppose that $U_t|_{t=0} = \theta(x)$. If we are able to find the function $\theta(x)$, then our problem reduces to the ordinary Cauchy problem, whose solution is

$$U(x,t) = \frac{1}{2\pi^{\frac{n-1}{2}}\Gamma(\lambda+1)} \int [t^2 - |x-\xi|^2]_+^\lambda \theta(\xi)\, d\xi\Big|_{\lambda=-\frac{n-1}{2}}.$$

Writing this solution for $|x| = t$, we obtain

$$U(x,|x|) \equiv \varphi(x) = \frac{1}{2\pi^{\frac{n-1}{2}}\Gamma(\lambda+1)} \int [2(x,\xi) - |\xi|^2]_+^\lambda \theta(\xi)\, d\xi\Big|_{\lambda=-\frac{n-1}{2}}.$$

The last equality can be transformed into the form

$$\varphi(x) = \frac{1}{|x|}\int_{(x,z)=|z|^2} \frac{dS_z}{|z|^{n-2}} \left[\frac{1}{2\pi^{\frac{n-1}{2}}\Gamma(\lambda+1)} \times \int_0^{2|z|} (2|z|-r)^\lambda r^{\frac{n-1}{2}} \theta\left(\frac{z}{|z|}r\right)\, dr\right]_{\lambda=-\frac{n-1}{2}}.$$

We have again arrived at the integral equation (1) for the function standing in square brackets. Let the number of spatial variables n be odd. Then our integral equation assumes the form

$$\varphi(x) = \frac{1}{|x|}\int_{(x,z)=|z|^2} \frac{dS_z}{|z|^{n-2}} \left\{\frac{1}{\pi^{\frac{n-1}{2}}} \frac{\partial^{\frac{n-3}{2}}}{\partial |z|^{\frac{n-3}{2}}} \left(|z|^{\frac{n-1}{2}}\theta(2z)\right)\right\}.$$

Solving it, we find

$$\frac{\partial^{\frac{n-3}{2}}}{\partial |z|^{\frac{n-3}{2}}}\left[|z|^{\frac{n-1}{2}}\theta(2z)\right] = \frac{1}{(-4\pi)^{\frac{n-1}{2}}} \frac{\partial^{n-1}}{\partial |z|^{n-1}} \int_{(x,z)=|z|^2} \varphi(x)\, dS_x.$$

Fig. 1

Figure 1: Fig. 1

Hence

$$\theta(2z) = \frac{1}{(-4\pi)^{\frac{n-1}{2}}} \frac{1}{|z|^{\frac{n-1}{2}}} \frac{\partial^{\frac{n+1}{2}}}{\partial |z|^{\frac{n+1}{2}}} \int_{(x,z)=|z|^2} \varphi(x)\, dS_x + \sum_{k=0}^{\left[\frac{n-4}{2}\right]} C_k\left(\frac{z}{|z|}\right) |z|^{-2-k}, \tag{15}$$

where $C_k(\omega)$ are arbitrary functions of the unit vector ω.

For uniqueness of the solution of problem II one may require sufficiently rapid decay of $\theta(z)$ at infinity (for example, $\theta(z) \in L_2$—then all $C_k(\omega) = 0$); or else maximal smoothness of $U(x,t)$ in a neighborhood of the cone $|x|^2 = t^2$, i.e. the least singularity of $\theta(z)$ at the origin—then as $C_k(z/|z|)$ we must take the first coefficients of the Taylor expansion of the integral

$$\frac{1}{(-4\pi)^{\frac{n-1}{2}}} \frac{\partial^{\frac{n+1}{2}}}{\partial |z|^{\frac{n+1}{2}}} \int_{(x,z)=|z|^2} \varphi(x)\, dS_x$$

with respect to powers of $|z|$.

We indicate the domain of dependence in our problem: the best-decaying term (i.e. $\theta(z)$, if all $C_k = 0$) depends on the values of $\varphi(x)$ at those points that lie on the intersection of the initial characteristic cone with the cone having its vertex at the point z (see Fig. 1).

Fig. 1.

Solving problem II in the even-dimensional case in a completely analogous way, we find that

$$\theta(z) = \theta_0(z) + \sum_{k=0}^{\frac{n-4}{2}} C_k\left(\frac{z}{|z|}\right) |z|^{-k-2},$$

where $C_k(z/|z|)$ are arbitrary functions of the unit vector, and

$$\theta_0(z) = B_n \frac{1}{|z|} \int dy\, \varphi\left(\frac{y}{2}\right) \int_0^1 \left| \frac{(z,y)}{|z|} - \rho |z| \right|^{\lambda} (1-\rho)^{\frac{n-5}{2}}\, d\rho \Bigg|_{\lambda=-n}.$$

The function $\theta_0(z)$ gives a solution possessing maximal smoothness in a neighborhood of the cone $|x|^2 = t^2$. It has the asymptotic form

$$\theta_0(z) = \sum_{k=0}^{\frac{n-4}{2}} D_k\left(\frac{z}{|z|}\right)|z|^{-k-2} + O\Big(|z|^{\frac{-2-\ln}{2}}\Big)\,.$$

Choosing $C_k(z/|z|) = -D_k(z/|z|)$, we obtain the solution that decreases at infinity most rapidly. We note that the domain of dependence for such a solution is the entire initial cone, and not the part of it lying inside the cone with vertex at the point z.

In exactly the same way one may consider the problems obtained from problem II if in it condition (14) is replaced by the conditions $U_t|_{t=0} = 0$ or $U|_{t=-|x|} = \varphi_1(x)$.

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Note: Figure translations are in progress. See original paper for figures.