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MATHEMATICAL PHYSICS

E. I. KIM and B. B. BAIMUKHANOV

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Abstract

Full Text

MATHEMATICAL PHYSICS

E. I. KIM and B. B. BAIMUKHANOV

ON THE DISTRIBUTION OF TEMPERATURE IN A PIECEWISE-HOMOGENEOUS SEMI-INFINITE PLATE

(Presented by Academician I. M. Vinogradov, 4 V 1961)

1. Find a continuous function $u(x, y, t)$ in the domain D $(x \geq 0;\ -\infty < y < +\infty;\ 0 \leq t \leq t_0)$, satisfying the equation

$$\frac{\partial u}{\partial t} = a^2(y)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (x > 0,\ y \neq 0,\ 0 < t < t_0), \tag{1}$$

where $a^2(y) = a_1^2$ if $y < 0$; $a^2(y) = a_2^2$ if $y > 0$, the initial condition

$$u(x, y, t)\big|_{t=0} = f(x, y), \tag{2}$$

the boundary condition

$$u(x, y, t)\big|_{x=0} = \varphi(y, t) \tag{3}$$

and the conjugation condition

$$u(x, -0, t) = u(x, +0, t), \qquad k_1 \frac{\partial u(x, -0, t)}{\partial y} = k_2 \frac{\partial u(x, -0, t)}{\partial y}, \tag{4}$$

where k_1, k_2 are positive constants. We shall seek the solution in the class of functions satisfying the inequality

$$\max_{0 \leq t \leq t_0} |u(x, y, t)| < M_0 e^{\delta^2 r^2}, \tag{5}$$

where M_0, δ are constants; $r = \sqrt{x^2 + y^2}$; t_0 is a constant satisfying the inequality

$$0 < t_0 < \frac{1}{4a_0^2\delta^2}, \qquad a_0 = \max(a_1, a_2). \tag{6}$$

For the continuity of the solution in the domain D and for condition (5), we impose on the functions $\varphi(y,t)$ and $f(x,y)$ the following restrictions:

1) The function $\varphi(y,t)$ is continuous in both arguments in the domain g $(-\infty < y < +\infty;\ 0 \leq t \leq t_0)$, the derivative with respect to t exists, and the first derivative with respect to y may undergo a discontinuity of the first kind along the x-axis. In addition, the inequalities

$$|\varphi(y,t)|,\ |\varphi'_y(y,t)|,\ |\varphi'_t(y,t)| < Me^{\delta^2 y^2}; \tag{7}$$

$$|\varphi'_y(y_1,t_1) - \varphi'_y(y_2,t_2)| < Me^{\delta^2 y_0^2}\left(|y_1 - y_2|^\alpha + |t_1 - t_2|^\alpha\right), \tag{8}$$

hold, where y_1 and y_2 have the same signs;

$$|\varphi'_t(y_1,t_1) - \varphi'_t(y_2,t_2)| < Me^{\delta^2 y_0^2}\left(|y_1 - y_2|^\alpha + |t_1 - t_2|^\alpha\right), \tag{9}$$

where M is a constant, $0 < \alpha \leq 1$, and $y_0 = \max(|y_1|, |y_2|)$.

2) The function $f(x,y)$ is continuous in both arguments in the domain G $(0 \leq x < +\infty,\ -\infty < y < +\infty)$, the derivative with respect to x exists, and the derivative with respect to y may undergo a discontinuity of the first kind along the x-axis. In addition, the inequalities

$$|f(x,y)|,\ |f'_x(x,y)|,\ |f'_y(x,y)| < Me^{\delta^2 r^2}, \tag{10}$$

$$|f'_x(x_1,y) - f'_x(x_2,y)|,\ |f'_y(x_1,y) - f'_y(x_2,y)| < Me^{\delta^2 r_0^2}|x_1 - x_2|^\alpha, \tag{11}$$

hold, where $r_0^2 = \max(x_1^2 + y^2;\ x_2^2 + y^2)$.

3) $\varphi(y,t)$ and $f(x,y)$ have the following relation:

$$\varphi(y,0) = f(0,y). \tag{12}$$

2. We shall seek the solution of the posed problem in the form:

for $y < 0$:

$$\begin{aligned} u(x,y,t) = {} & \int_0^t d\tau \int_{-\infty}^{+\infty} \frac{x\varphi(\eta,\tau)}{4\pi a_1^2 (t-\tau)^2} \exp\left[-\frac{x^2 + (y-\eta)^2}{4a_1^2(t-\tau)}\right] d\eta \\ & + \int_0^t d\tau \int_{-\infty}^{+\infty} \frac{\psi_1(\xi,\tau)}{2\pi(t-\tau)} \exp\left[-\frac{(x-\xi)^2 + y^2}{4a_1^2(t-\tau)}\right] d\xi \\ & + \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} \frac{f_0(\xi,\eta)}{4\pi a_1^2 t} \exp\left[-\frac{(x-\xi)^2 + (y-\eta)^2}{4a_1^2 t}\right] d\eta; \end{aligned} \tag{13}$$

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for $y > 0$:

$$u(x,y,t) = \int_0^t d\tau \int_{-\infty}^{+\infty} \frac{x\varphi(\eta,\tau)}{4\pi a_2^2(t-\tau)^2} \exp\left[-\frac{x^2+(y-\eta)^2}{4a_2^2(t-\tau)}\right] d\eta \\ + \int_0^t d\tau \int_{-\infty}^{+\infty} \frac{y\psi_2(\xi,\tau)}{4\pi a_2^2(t-\tau)^2} \exp\left[-\frac{(x-\xi)^2+y^2}{4a_2^2(t-\tau)}\right] d\xi \\ + \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} \frac{f_0(\xi,\eta)}{4\pi a_2^2 t} \exp\left[-\frac{(x-\xi)^2+(y-\eta)^2}{4a_2^2 t}\right] d\eta, \tag{14}$$

where $f_0(x,y) = f(x,y)$ for $x > 0$; $f_0(x,y) = -f(-x,y)$ for $x < 0$; $f_0(0,y) = 0$; if $\psi_1(x,t)$ and $\psi_2(x,t)$ are odd functions with respect to the first argument, then the function $u(x,y,t)$ satisfies equation (1), the initial condition (2), and the boundary condition (3), since it is expressed by heat potentials (^{2}).

It is necessary to choose the functions $\psi_1(x,t)$ and $\psi_2(x,t)$ from the class of odd functions with respect to the first argument so that the function $u(x,y,t)$ satisfies the conjugation conditions (4) and condition (5). It is not difficult to verify that if the functions $\psi_1(x,t)$ and $\psi_2(x,t)$ satisfy the conditions

$$|\psi_1(x,t)| < \frac{M}{\sqrt{t}} e^{\delta^2 x^2}, \qquad |\psi_2(x,t)| < M e^{\delta^2 x^2}, \tag{15}$$

then, provided conditions (6), (7), and (10) are fulfilled, the function $u(x,y,t)$ satisfies inequality (5).

3. We now determine the functions $\psi_1(x,t)$ and $\psi_2(x,t)$ in formulas (13) and (14). In doing so we use the properties of the heat potential (^{2}). Using the first condition (4), we have

$$\psi_2(x,t) = \int_0^t d\tau \int_{-\infty}^{+\infty} \frac{\psi_1(\xi,\tau)}{2\pi(t-\tau)} \exp\left[-\frac{(x-\xi)^2}{4a_1^2(t-\tau)}\right] d\xi \\ + \int_0^t d\tau \int_{-\infty}^{+\infty} \frac{x\varphi(\eta,\tau)}{4\pi a_1^2(t-\tau)^2} \exp\left[-\frac{x^2+\eta^2}{4a_1^2(t-\tau)}\right] d\eta \\ - \int_0^t d\tau \int_{-\infty}^{+\infty} \frac{x\varphi(\eta,\tau)}{4\pi a_2^2(t-\tau)^2} \exp\left[-\frac{x^2+\eta^2}{4a_2^2(t-\tau)}\right] d\eta \\ + \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} \frac{f_0(\xi,\eta)}{4\pi a_1^2 t} \exp\left[-\frac{(x-\xi)^2+\eta^2}{4a_1^2 t}\right] d\eta \\ - \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} \frac{f_0(\xi,\eta)}{4\pi a_2^2 t} \exp\left[-\frac{(x-\xi)^2+\eta^2}{4a_2^2 t}\right] d\eta. \tag{16}$$

Applying the second condition (4), we obtain

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$$
\begin{aligned}
k_1\psi_1(x,t) = k_2 \lim_{y\to+0}\frac{\partial}{\partial y}\int_0^t d\tau \int_{-\infty}^{+\infty} \frac{y\psi_2(\xi,\tau)}{4\pi a_2^2(t-\tau)^2}\exp\left[-\frac{(x-\xi)^2+y^2}{4a_2^2(t-\tau)}\right] d\xi + \\
+ k_2\int_0^t d\tau\int_{-\infty}^{+\infty}\frac{x\varphi'_\eta(\eta,\tau)}{4\pi a_2^2(t-\tau)}\exp\left[-\frac{x^2+\eta^2}{4a_2^2(t-\tau)}\right] d\eta \\
- k_1\int_0^t d\tau\int_{-\infty}^{+\infty}\frac{x\varphi'_\eta(\eta,\tau)}{4\pi a_1^2(t-\tau)^2}\exp\left[-\frac{x^2+\eta^2}{4a_1^2(t-\tau)}\right] d\eta \\
+ k_2\int_{-\infty}^{+\infty} d\xi\int_{-\infty}^{+\infty}\frac{f'_{0\eta}(\xi,\eta)}{4\pi a_2^2 t}\exp\left[-\frac{(x-\xi)^2+\eta^2}{4a_2^2 t}\right] d\eta \\
- k_1\int_{-\infty}^{+\infty} d\xi\int_{-\infty}^{+\infty}\frac{f'_{0\eta}(\xi,\eta)}{4\pi a_1^2 t}\exp\left[-\frac{(x-\xi)^2+\eta^2}{4a_1^2 t}\right] d\eta.
\end{aligned}
\tag{17}
$$

To compute the limit in the second integral equation, we state the following lemma.

Lemma. If the function $\psi_1(x,t)$ satisfies the condition

$$
|\psi_1(x_1,t)-\psi_1(x_2,t)| < \frac{M}{\sqrt{t}}e^{\delta^2 d^2}|x_1-x_2|^\alpha, \tag{18}
$$

where $d=\max(|x_1|,|x_2|)$, $0<\alpha\leqslant 1$, separately on the intervals $(-\infty,0)$ and $(0,+\infty)$, and may have a discontinuity of the first kind at $x=0$, then

$$
\begin{aligned}
k_2\lim_{y\to+0}\frac{\partial}{\partial y}\int_0^t\frac{y\,d\tau}{4\pi a_2^2(t-\tau)^2}\int_{-\infty}^{+\infty}\exp\left[-\frac{(x-\xi)^2+y^2}{4a_2^2(t-\tau)}\right]\int_0^\tau d\tau_1\int_{-\infty}^{+\infty}\frac{\psi_1(\xi_1,\tau_1)}{2\pi(\tau-\tau_1)}\times \\
\times\exp\left[-\frac{(\xi-\xi_1)^2}{4a_1^2(\tau-\tau_1)}\right] d\xi_1 d\xi = -\frac{a_1k_2}{a_2}\psi_1(x,t)+\frac{a_1(a_1^2-a_2^2)k_2}{2\pi^{3/2}a_2}\int_0^t\frac{d\tau}{(t-\tau)^{3/2}}\times \\
\times\int_{-\infty}^{+\infty}\psi_1(\xi,\tau)\int_0^{+\infty}\rho(z)\left[1-\frac{(x-\xi)^2}{2a^2(z)(t-\tau)}\right]\exp\left[-\frac{(x-\xi)^2}{4a^2(z)(t-\tau)}\right] dz\,d\xi,
\end{aligned}
\tag{19}
$$

where

$$
\rho(z)=(z^2+a_1^2)^{-3/2}(z^2+a_2^2)^{-1/2},\qquad a^2(z)=a_2^2\,\frac{z^2+a_1^2}{z^2+a_2^2}.
$$

To prove the lemma, we interchange the integrals. After computing the inner integrals, we differentiate with respect to y under the integral sign. Then, by computing the limit and using the conditions of the lemma, we obtain formula (19).

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4. From the system (16) and (17), let us eliminate the unknown function $\psi_2(y,t)$. In doing so, using (19) and (7)–(12), we obtain the following integral equation with respect to $\psi_1(x,t)$:

$$\begin{aligned}\psi_1(x,t) = \lambda \int_0^t \frac{d\tau}{(t-\tau)^{3/2}} \int_{-\infty}^{+\infty} \psi_1(\xi,\tau) \int_{-\infty}^{+\infty} \rho(z) \left[1 - \frac{(x-\xi)^2}{2a^2(z)(t-\tau)}\right] \times \\ \times \exp\left[-\frac{(x-\xi)^2}{4a^2(z)(t-\tau)}\right] dz\, d\xi + F(x,t),\end{aligned} \tag{20}$$

where

$$\begin{aligned}F(x,t) = {} & \frac{k_2}{2\pi a_2(a_1k_2+a_2k_1)} \int_0^t \frac{d\tau}{(t-\tau)^2} \int_0^{+\infty} x\varphi'_\eta(\eta,\tau) \exp\left[-\frac{x^2+\eta^2}{4a_2^2(t-\tau)}\right] d\eta \\ & - \frac{a_2k_1}{4\pi a_1^2(a_1k_2+a_2k_1)} \int_0^t \frac{d\tau}{(t-\tau)^2} \int_{-\infty}^{+\infty} x\varphi'_\eta(\eta,\tau) \exp\left[-\frac{x^2+\eta^2}{4a_1^2(t-\tau)}\right] d\eta +\end{aligned}$$

$$\begin{aligned}& + \frac{(a_2^2-a_1^2)\,k_2}{2\pi^{3/2}a_1a_2^2(a_1k_2+a_2k_1)} \int_0^t \frac{d\tau}{(t-\tau)^{3/2}} \int_{-\infty}^{+\infty} x\varphi'_\eta(\eta,\tau) \exp\left[-\frac{\eta^2}{4a_1^2(t-\tau)}\right] \times \\ & \times \int_0^{+\infty} \frac{(z^2+a_2^2)^{1/2}}{(z^2+a_1^2)^{3/2}} \exp\left[-\frac{(\eta z)^2}{4a_1^2a_2^2(t-\tau)} - \frac{x^2}{4a_1^2(t-\tau)}\frac{z^2+a_1^2}{z^2+a_2^2}\right] dz\, d\eta \\ & - \frac{a_2^2k_2}{2\pi^{3/2}a_1^3(a_1k_2+a_2k_1)} \int_0^t \frac{d\tau}{(t-\tau)^2} \int_{-\infty}^{+\infty} x\varphi'_\eta(\eta,\tau) \exp\left[-\frac{\eta^2}{4a_1^2(t-\tau)}\right] \times \\ & \times \int_0^{+\infty} \left(\frac{\eta^2+4a_1^2(t-\tau)z^2}{\eta^2+4a_2^2(t-\tau)z^2}\right)^{3/2} \exp\left[-z^2 - \frac{x^2}{4a_1^2(t-\tau)}\frac{\eta^2+4a_1^2(t-\tau)z^2}{\eta^2+4a_2^2(t-\tau)z^2}\right] \operatorname{sign}\eta\, dz\, d\eta \\ & + \frac{k_2}{2\pi a_2(a_1k_2+a_2k_1)t} \int_{-\infty}^{+\infty} d\xi \int_0^{+\infty} f'_{0\eta}(\xi,\eta) \exp\left[-\frac{(x-\xi)^2+\eta^2}{4a_2^2t}\right] d\eta \\ & - \frac{a_2k_1}{4\pi a_1^2(a_1k_2+a_2k_1)t} \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} f'_{0\eta}(\xi,\eta) \exp\left[-\frac{(x-\xi)^2+\eta^2}{4a_1^2t}\right] d\eta \\ & + \frac{(a_1^2-a_2^2)k_2}{4\pi^{3/2}a_1a_2^2(a_1k_2+a_2k_1)t^{3/2}} \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} (x-\xi) f'_{0\xi}(\xi,\eta) \exp\left[-\frac{\eta^2}{4a_1^2t}\right] \times \\ & \times \int_0^{+\infty} \frac{(z^2+a_2^2)^{1/2}}{(z^2+a_1^2)^{3/2}} \exp\left[-\frac{(\eta z)^2}{4a_1^2a_2^2t} - \frac{(x-\xi)^2}{4a_2^2t}\frac{z^2+a_2^2}{z^2+a_1^2}\right] dz\, d\eta \\ & - \frac{k_2}{2\pi^{3/2}a_1(a_1k_2+a_2k_1)t} \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} f'_{0\eta}(\xi,\eta) \exp\left[-\frac{\eta^2}{4a_1^2t}\right] \times \\ & \times \int_0^{+\infty} \sqrt{\frac{\eta^2+4a_1^2tz^2}{\eta^2+4a_2^2tz^2}} \exp\left[-z^2 - \frac{(x-\xi)^2}{4a_1^2t}\frac{\eta^2+4a_1^2tz^2}{\eta^2+4a_2^2tz^2}\right] \operatorname{sign}\eta\, dz\, d\eta.\end{aligned}$$

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We shall seek the solution of equation (20) in the class of odd functions satisfying inequality (15) and the conditions of the lemma. By direct verification one can establish that the free term $F(x,t)$ is an odd function satisfying the first inequality (15) and all the conditions of the lemma.

Thus, if equation (20) has a solution, then it is the desired one. The existence of a solution of an equation of this kind was proved by one of the authors [1]. The solution is expressed by the following formula

$$\psi_1(x,t) = F(x,t) + \lambda \int_0^t d\tau \int_{-\infty}^{+\infty} R(x-\xi, t-\tau; \lambda) F(\xi,\tau)\, d\xi,$$

where

$$R(x-\xi, t-\tau; \lambda) = \frac{(1+\nu)^2}{(t-\tau)^{1/2}} \int_0^{+\infty} \frac{z^2 \rho(z)}{\nu^2 z^2 + a_2^2} \left[1 - \frac{(x-\xi)^2}{2a^2(z)(t-\tau)}\right] \times$$

$$\times \exp\left[-\frac{(x-\xi)^2}{2a^2(z)(t-\tau)}\right] dz, \qquad \lambda = \frac{(a_1^2 - a_2^2) a_1 k_2}{2\pi^{3/2}(a_1 k_2 + a_2 k_1)}, \qquad \nu = \frac{a_1^2 - a_2^2 - 2\pi^{3/2}\lambda}{2\pi^{3/2}\lambda}.$$

After determining $\psi_1(x,t)$, we determine $\psi_2(x,t)$ by formula (16), and thereby the problem is completely solved.

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Note: Figure translations are in progress. See original paper for figures.

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