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Abstract

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MATHEMATICS

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SIMPLE SUBGROUPS OF SIMPLY CONNECTED REAL SIMPLE LIE GROUPS

(Presented by Academician I. G. Petrovskii, 26 XI 1960)

The simple subalgebras of the real forms of the classical Lie algebras were found by F. I. Karpelevich (1). To a subalgebra G^* of a Lie algebra G there corresponds, in the simply connected group $\widetilde{\mathfrak{G}}$ corresponding to the algebra G, a certain connected subgroup $\mathfrak{G}^*$, locally isomorphic to the simply connected group $\widetilde{\mathfrak{G}}^*$ with Lie algebra G^*. In the present paper we set forth methods for computing the group $\mathfrak{G}^*$, i.e. the kernel of the corresponding homomorphism $\widetilde{\mathfrak{G}}^* \to \mathfrak{G}^*$. Everywhere below, unless otherwise stated, G^* and G are real forms of the classical Lie algebras, considered in their canonical linear representation.

1. The imbedding $G^* \subset G$ is given (1) by means of a linear representation φ of the algebra G^*, $\varphi : G^* \to G$, to which there correspond, first, a homomorphism $\widetilde{\varphi} : \widetilde{\mathfrak{G}}^* \to \widetilde{\mathfrak{G}}$, whose kernel $\ker \widetilde{\varphi}$ we shall seek, and, secondly, a linear representation $\hat{\varphi} : \widetilde{\mathfrak{G}}^* \to \mathfrak{GL}$, whose kernel $\ker \hat{\varphi}$ can be found from the formulas of (2). If the linear group corresponding to G is simply connected (i.e. coincides with $\widetilde{\mathfrak{G}}$), then $\ker \widetilde{\varphi} = \ker \hat{\varphi}$. This condition is satisfied by $G = J$ or C_n^l, which will not be considered below.

Let us denote the characteristic subalgebras of the algebras G and G^* respectively by P and P^*. For them there are decompositions: $P = K + V$, $P^* = K^* + V^*$, where K, K^* are semisimple compact algebras, each of which consists of at most two simple components, while V, V^* are commutative algebras of dimension 0 or 1. According to (1), one may assume that $\varphi(P^*) \subset P$. It is easy to see that then $\varphi(K^*) \subset K$.

The following will be useful in what follows:

Lemma 1.1). If $\varphi(P^*) \subset K$, $K = K' + K''$ is the decomposition into (nontrivial) simple components, and $\psi_1, \dots, \psi_k$ are the irreducible components of the linear representation of the subalgebra P^* induced by the representation φ, then for every $j = 1, \dots, k$ either $\varphi_j(P^*) \subset K'$, or $\varphi_j(P^*) \subset K''$, and thus on P^* the representation $\varphi = \varphi' + \varphi''$, where $\varphi'(P^*) \subset K'$, $\varphi''(P^*) \subset K''$.

2) If the algebras B_n^2 and D_n^2 are excluded from consideration, then for one-dimensional V there corresponds to the subalgebra $K \subset G$ a simply connected linear group.

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Theorem 1. If the linearizing normal subgroup [2] of the group $\widetilde{\mathfrak{G}}$ and the center of $\widetilde{\mathfrak{G}}^*$ are infinite cyclic groups, then all subgroups $\widetilde{\mathfrak{G}}$ locally isomorphic to $\widetilde{\mathfrak{G}}^*$ are either simply connected or isomorphic to linear groups.

Here G and G^* are arbitrary real simple algebras, not only classical ones. The hypotheses of the theorem are satisfied, for example, when $G = A_n^l, B_n^2, D_n^2, IC_n$ or JD_n, and $G^* = A_n^l$ for o.n.d. $(l, n-l+1) = 1$, IC_n or JD_n for odd n [2,3].

Theorem 2. If to the subalgebra $K \subset G$ there corresponds a simply connected linear group, then either $\ker\widetilde{\varphi} = \ker\hat{\varphi}$, or $\ker\widetilde{\varphi} = \ker\hat{\varphi} \cap \vartheta$, where ϑ is a sub-

the group of the center of the group $\widetilde{G^*}$, consisting of all elements of finite order. In particular, if the characteristic subalgebra of the algebra G^* is semisimple ($P^* = K^*$), then $\ker\widetilde{\varphi} = \ker\widehat{\varphi}$.

The conditions of the theorem are fulfilled in those cases when G is one of the algebras of type A_n^l, IC_n, or JD_n [2]. The last assertion of the theorem solves our problem for $G^* = B_n^{2l}$, D_n^l, I_n, or J_n [3]. It remains to consider the case when $G^* = A_{n-1}^l$, IC_n, or JD_n. Since all these algebras are of the first category, their embedding in G is given by a linear representation $\varphi = m_1\varphi_1 + \dots + m_k\varphi_k$ ($\varphi_1, \dots, \varphi_k$ are inequivalent irreducible representations; $m_1, \dots, m_k$ are their multiplicities) and by the numbers $q_1, \dots, q_k$ [1]. We shall specify the representation φ_i by its highest weight, whose coordinates in the Cartan-Weyl system are nonnegative integers $f_1 \geq \dots \geq f_n \geq 0$ [4]. The number $v_i = f_1 + \dots + f_n$ is called the rank of the representation φ_i.

First let $G^* = A_{n-1}^l$. Since in this case $P^* = A_{l-1} + A_{n-l-1} + V$, the representation of the subalgebra P^* induced by the representation φ_i decomposes into irreducible components of the form $\psi'_{i_1} \times \psi''_{i_1} + \psi'_{i_2} \times \psi''_{i_2} + \cdots$, where ψ'_{ij} is a representation of the full linear algebra $L(l)$; ψ''_{ij} is a representation of $L(n-l)$; and $\psi'_{ij} \times \psi''_{ij}$ is their tensor product. Denote by N_{ij} the dimension of the representation $\psi'_{ij} \times \psi''_{ij}$, and by v_{ij} the rank of the representation ψ'_{ij}.

The formula $\ker\widetilde{\varphi} = \ker\widehat{\varphi}$ holds if and only if the following condition is fulfilled:

$$\sum_{i=1}^{k} q_i \sum_j (lv_i - nv_{ij})N_{ij} = 0,$$

where the inner summation is over all j for which v_{ij} is even.

Now let $G^* = IC_n$ or JD_n. For these algebras $P^* = V + A_n$, and the decomposition of the representation of the subalgebra P^* induced by the representation φ_i has the form $\psi_{i1} + \psi_{i2} + \cdots$, where ψ_{ij} is a representation of the algebra $L(n)$ of dimension N_{ij} and rank v_{ij} (not necessarily positive, since the highest weight may here also have negative coordinates), and moreover $v_{ij} \equiv v_i \pmod 2$ for all j.

The formula $\ker\widetilde{\varphi} = \ker\widehat{\varphi}$ holds if and only if the following condition is fulfilled:

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$$\sum_{i=1}^{k} q_i \sum_j v_{ij} N_{ij} = 0,$$

where the inner summation is over all j for which $v_{ij} \equiv v_i \pmod 4$.

It remains for us to consider the case when the subalgebra $K \subset G$ corresponds to a non-simply connected linear group. According to the second assertion of the lemma, now $V = 0$, and consequently $\varphi(P^*) \subset K$ (we exclude the case $G = B_n^2$ or D_n^2 in order not to encumber the exposition). Therefore, if the algebra K is simple, then, as follows easily from Lemma 1 of ([3]), the question reduces to computing the kernel of a homomorphism into a simple compact group, i.e. to a problem solved in ([5]) (this condition for K was verified for the case $G = I_n$). If, however, $K = K' + K''$ (which corresponds to the case $G = B_n^{2l}$ or D_n^l, where K' and K'' are orthogonal algebras), then, in order to apply the results of ([5]), it is necessary to find, according to the lemma, φ' and φ''. Let σ be the characteristic automorphism selecting the algebra G^* from its compact form R (σ acts in R and $P^* \subset R$; see, for example, ([1])). Denote by $\mathcal{R}$ the simply connected compact group for the Lie algebra R, and by $\mathfrak{P}^* \subset \mathcal{R}$ the connected subgroup,

corresponding subalgebra P^*. We extend the action σ to $\mathcal{R}$. Let $\overline{\mathcal{R}}$ be the smallest group satisfying the following requirements: $\overline{\mathcal{R}}$ contains $\mathcal{R}$ and an element s such that $s^{-1} r s = \sigma(r)$ for all $r \in \mathcal{R}$ (if the automorphism σ is inner, then $\overline{\mathcal{R}} = \mathcal{R}$). The embedding $G^* \subset G$ is given, in the general case, by a linear representation of the algebra G^* (and hence also of R)

$$\varphi = m_1\varphi_1 + \dots + m_k\varphi_k + m_{k+1}(\varphi'_{k+1} + \varphi''_{k+1}) + \dots + m_p(\varphi'_p + \varphi''_p),$$

where $\varphi_h\sigma \sim \varphi_h$ $(h = 1, \dots, k)$, $\varphi'_i\sigma \sim \varphi''_i$ $(i = k+1, \dots, p)$, and by a set of numbers $q_1, \dots, q_k$, or $l_1, \dots, l_k$, where $q_h = m_h - 2l_h$ [1]. The representation φ_h, by virtue of the condition $\varphi_h\sigma \sim \varphi_h$, can be extended to the whole group $\overline{\mathcal{R}}$. Let $\mathcal{P}^*$ be the smallest subgroup of $\overline{\mathcal{R}}$ containing $\mathcal{P}^*$ and s, and let $\psi_1 + \psi_2 + \cdots$ be the decomposition into irreducible components of the representation of the group $\overline{\mathcal{P}^*}$ induced by the representation φ_h. Each ψ_j, when considered on $\mathcal{P}^*$, either remains irreducible or splits into two components. Let χ_j be the character of the representation ψ_j, and N_j its dimension; then for all j

$$\chi_j(s) = \pm\lambda N_j,$$

where λ is a certain number. Put $\varphi'_h = \sum' \psi_j$, where the summation extends over all j for which $\chi_j(s) = \lambda N_j$. The remaining ψ_j make up φ''_h.* After φ'_h, φ''_h have been found, one can find φ', φ'':

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$$\varphi' = \sum_{h=1}^{k} [(m_h - l_h)\varphi'_h + l_h \varphi''_h] + \sum_{i=k+1}^{p} m_i \varphi'_i,$$

$$\varphi'' = \sum_{h=1}^{k} [l_h \varphi'_h + (m_h - l_h)\varphi''_h] + \sum_{i=k+1}^{p} m_i \varphi''_i.$$

2. In the actual use of the results obtained, in a number of cases one has to find the irreducible components of the representation of the characteristic subalgebra P^* induced by a certain irreducible representation of the algebra G^*.

Let us consider, on the full linear group $GL(r)$, the function $\{f_1, \dots, f_u\}$, defined, for given nonnegative integers $f_1, \dots, f_u$, by the formula

$$\{f_1, \dots, f_u\} = |p_{l_i - u + 1},\ p_{l_i - u + 2}, \dots,\ p_{l_i}|.$$

Here on the right-hand side is written the i-th row of a determinant of order u, where $l_1 = f_1 + u - 1,\ l_2 = f_2 + u - 2, \dots,\ l_u = f_u$, and the quantities $p_0, p_1, p_2, \dots$ are determined from the equalities

$$\frac{1}{|E - zA|} = p_0 + p_1 z + p_2 z^2 + \cdots, \qquad p_{-k} = 0 \quad \text{for } k = 1, 2, \dots;$$

A is an element of the group $GL(r)$ at which the value of $\{f_1, \dots, f_u\}$ is computed. If $u = r$ and $f_1 \geq \dots \geq f_r \geq 0$, then, as is known [(4)], the function $\{f_1, \dots, f_r\}$ is equal to the character $\chi(f_1, \dots, f_r)$ of the irreducible representation of the group $GL(r)$ with highest weight $f_1, \dots, f_r$.

It is easy to verify the following properties of the function $\{f_1, \dots, f_u\}$ [(4)]:
1) $\{f_1, \dots, f_{u-1}, 0\} = \{f_1, \dots, f_{u-1}\}$;
2) if $f_u < 0$ or if $f_1 \geq \dots \geq f_u \geq 0$ and $u > r$, then $\{f_1, \dots, f_u\} = 0$;
3) if $f_{i+1} - f_i = 1$, then $\{\dots, f_i, f_{i+1}, \dots\} = 0$, while if $f_{i+1} - f_i > 1$, then

$$\{\dots, f_i, f_{i+1}, \dots\} = \{\dots, f_{i+1} - 1, f_i + 1, \dots\}.$$

With the help of the last property one can reduce such a function to a form for which $f_1 \geq \dots \geq f_u$, and then it turns out that it is equal

* In order to preserve formula (1) of the paper for the signature of the algebra G, one would have to interchange φ'_h and φ''_h in those cases where the dimension of φ'_h turns out to be less than the dimension of φ''_h.

either 0, or equal, perhaps with a minus sign, to some primitive character of the group $\mathfrak{GL}(r)$.

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Introduce the operators $\xi_i,\ i = 1, 2, \dots$, acting on the function $\{f_1, \dots, f_u\}$ by the formula

$$\xi_i\{f_1, \dots, f_i, \dots, f_u\} = \{f_1, \dots, f_i - 1, \dots, f_u\}.$$

Consider the subgroup $\mathfrak{GL}(l) \cdot \mathfrak{GL}(n-l)$ of the group $\mathfrak{GL}(n)$, consisting of matrices $A = A_1 + A_2$, where $A_1,\ A_2$ are diagonal blocks of the matrix $A \in \mathfrak{GL}(n)$ of dimensions l and $n-l$, respectively. The character $\chi(f_1, \dots, f_n)$, $f_1 \geq \dots \geq f_n \geq 0$, of a certain irreducible representation φ of the group $\mathfrak{GL}(n)$ on this subgroup can be represented in the form

$$\chi(f_1, \dots, f_n) = \sum_{f_1 \geq s_1 \geq \dots \geq s_n \geq 0} (\nabla(s_1, \dots, s_n)\{f_1, \dots, f_n\}) \cdot \{s_1, \dots, s_n\},$$

where the value $\{s_1, \dots, s_n\}$ is computed at the point $A_2 \in \mathfrak{GL}(n-l)$, while the functions obtained as a result of the action of the operator

$$\nabla(s_1, \dots, s_n) = \frac{\left|\xi_i^{t_1} \dots \xi_i^{t_n}\right|}{\left|\xi_i^{n-1} \dots 1\right|}, \qquad t_1 = s_1 + n - 1, \dots, t_n = s_n,$$

on $\{f_1, \dots, f_n\}$, are computed at the point $A_1 \in \mathfrak{GL}(l)$. The expression obtained in this way can, according to what was said above, be reduced to the form

$$\sum_q c_q \chi'_q \chi''_q,$$

where c_q are integers, and χ'_q and χ''_q are primitive characters of $\mathfrak{GL}(l)$ and $\mathfrak{GL}(n-l)$, respectively. Since $\chi'_q \cdot \chi''_q$ is the character of a certain irreducible representation of the group $\mathfrak{GL}(l) \cdot \mathfrak{GL}(n-l)$, and the expression of the character of a reducible representation of the group as a linear combination of primitive characters is unique, all $c_q \geq 0$, and the equality

$$\chi(f_1, \dots, f_n) = \sum_q c_q \chi'_q \chi''_q$$

gives the decomposition into irreducible components of the representation of the group $\mathfrak{GL}(l) \cdot \mathfrak{GL}(n-l)$ induced by the representation φ of the group $\mathfrak{GL}(n)$.

Putting

$$\pi_k = \sum_{i_1 + \dots + i_n = k,\ i \geq 0} \xi_1^{i_1} \xi_2^{i_2} \dots \xi_n^{i_n},$$

one can express the operator $\nabla(s_1, \dots, s_n)$ through $\xi_1, \dots, \xi_n$ in an integral rational manner:

$$\nabla(s_1, \dots, s_n) = \left|\pi_{t_i - n + 1}, \dots, \pi_{t_i}\right|.$$

The obtained decomposition of the representation φ obviously solves the problem posed at the beginning of this section for the algebras A^l_{n-1}. The analogous problem for the algebras I_n (for which P^* is the orthogonal algebra) was solved in ([4,6]). With the aid of the methods of ([6]) and the results obtained here, one can find the decomposition of the representations of characteristic subalgebras also for the remaining classical real Lie algebras.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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