MATHEMATICS

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

M. ROSENBLATT-ROT

ON THE STRONG LAW OF LARGE NUMBERS FOR NONHOMOGENEOUS MARKOV CHAINS

(Presented by Academician A. N. Kolmogorov on 26 VII 1961)

Let α_i be the ergodicity coefficient of the i-th probabilistic transition function of a certain, in general nonstationary, Markov chain (see ([1],[2])), and denote

$$\eta_n = \max_{1\leq i\leq n-1}(1-\alpha_i), \qquad 1-\eta_n = O(n^{-\beta}) \quad (0\leq\beta<1).$$

Theorem 1. If a sequence of random variables ξ_i $(i \in I)$*, connected in a nonhomogeneous Markov chain with nonzero ergodicity coefficients $\alpha_i > 0$ $(i \in I)$, satisfies the condition

$$\sum_{n=1}^{\infty}\frac{\mathbf{D}\xi_n}{n^{2-\beta}} < +\infty,$$

then it obeys the strong law of large numbers.

If $\alpha_i > \rho > 0$ $(i \in I)$, this condition takes the form

$$\sum_{n=1}^{\infty}\frac{\mathbf{D}\xi_n}{n^{2}} < +\infty,$$

and it is best possible in the sense that if, for some sequence of nonnegative constants b_n, the series

$$\sum_{n=1}^{\infty}\frac{b_n}{n^2}$$

diverges, then one can construct a sequence of random variables ξ_n $(n \in I)$, connected in a Markov chain (not expressible as a sequence of independent random variables), with $\mathbf{D}\xi_n = b_n$, $\beta = 0$, which does not obey the strong law of large numbers.

Theorem 2. If, under the conditions of Theorem 1, $\mathbf{D}\xi_i \leq C < \infty$ $(i \in I)$, then the sequence ξ_i $(i \in I)$ obeys the strong law of large numbers.

Theorem 3. If in a discrete Markov chain with $\alpha_i > 0$ $(i \in I)$ the probability of the occurrence of event i in the k-th trial is equal to $p_k^{(i)}$, and $\mu^{(i)}$ denotes the number of occurrences of event i in the first n trials, then

$$\mathbf{P}\left\{\lim_{n\to\infty}\left(\frac{\mu^{(i)}}{n} - \frac{p_1^{(i)} + p_2^{(i)} + \cdots + p_n^{(i)}}{n}\right) = 0\right\} = 1.$$

Let R be the real line and let $\Omega = \{\omega_k\}$ be some finite or countable system of disjoint Borel sets on it, such that

$$\bigcup_k \omega_k = R.$$

The totality of all real random variables ξ

---

* I is the totality of all natural numbers, $\mathbf{M}$ is mathematical expectation, $\mathbf{D}$ is variance.

we shall partition into nonintersecting classes $\Lambda = \Lambda(\Omega)$, so that $\mathbf{P}\{|\xi| \in \omega_k\} = \varphi_\Lambda(k)$ depends only on Λ, and not on $\xi \in \Lambda$, for all k. All ξ contained in one and the same Λ will be called Ω-identically distributed. Obviously, if all random variables of some collection are identically distributed, then they are Ω-identically distributed for any system Ω. Let

$$\omega_k = \{k \le x < k+1\}, \qquad \Omega = \{\omega_k\} \quad (k \in I).$$

Theorem 4. For a sequence of Ω-identically distributed random variables $\xi_n \in \Lambda$ $(n \in I)$, connected into a nonstationary Markov chain with $\alpha_i > 0$ $(i \in I)$, the existence of some random variable $\xi \in \Lambda$ possessing a finite moment of order $1 + \beta$ is sufficient for the applicability of the strengthened law of large numbers.

If $\alpha_i > \rho > 0$ $(i \in I)$ (for example, in the case of independent ξ_n), for this it is sufficient that there exist some random variable $\xi \in \Lambda$ possessing a finite mathematical expectation.

Theorem 5. The existence of the mathematical expectation is a necessary and sufficient condition for the applicability of the strengthened law of large numbers to a sequence of random variables that are identically distributed and connected into a homogeneous Markov chain with ergodicity coefficient $\alpha > 0$.

Theorem 6. Let μ_i be the number of occurrences of event i in n consecutive trials according to the law of a homogeneous Markov chain with $\alpha > 0$, and let p_i be the probability of occurrence of event i in each of the trials; then

$$\mathbf{P}\left\{\lim_{n\to\infty}\frac{\mu_i}{n} = p_i\right\} = 1.$$

In the proof of Theorem 1 the following results are used.

Lemma 1. If the random variables ξ_i $(i \in I)$ are connected into a nonhomogeneous Markov chain with nonzero ergodicity coefficients $\alpha_i > 0$ $(i \in I)$ and have finite variances, then for any $\varepsilon > 0$ the inequality

$$\mathbf{P}\left\{\max_{1\le\Delta\le n}\sum_{k=1}^{s}|(\xi_k-\mathbf{M}\xi_k)|>\varepsilon\right\}\le\frac{n^{\beta}}{\varepsilon_1^2}\sum_{i=1}^{n}\mathbf{D}\xi_i,$$

holds, where ε_1 is a certain constant (depending on ε and on the chain).

If $\alpha_i > \rho > 0$ $(i \in I)$, this condition becomes

$$\mathbf{P}\left\{\max_{1\le s\le n}\left|\sum_{k=1}^{s}(\xi_k-\mathbf{M}\xi_k)\right|>\varepsilon\right\}\le\frac{1}{\varepsilon_1^2}\sum_{i=1}^{n}\mathbf{D}\xi_i.$$

Lemma 2. In order that a sequence of random variables ξ_i $(i \in I)$, connected into a Markov chain, satisfy the strengthened law of large numbers, it is necessary that for every $i > 0$ the condition

$$\sum_{n=1}^{\infty}\mathbf{P}\left\{|\xi_n-\mathbf{M}\xi_n|>\frac{\varepsilon n}{|\xi_{n-1}-\mathbf{M}\xi_{n-1}|}\le\varepsilon(n-1)\right\}<+\infty$$

be fulfilled.

Whatever $\varphi(n) = o(n)$ may be, this condition ceases to be necessary if in it εn is replaced by $\varphi(n)$.

Theorems 1-5 and Lemma 1 generalize the classical results of A. N. Kolmogorov (3-6), while Theorem 6 generalizes the classical Borel-Cantelli theorem; they are obtained from our results when $\rho = 1$, i.e. $\alpha_i = 1$ $(i \in I)$,

i.e., when the Markov chain degenerates into a sequence of independent random variables ((1), p. 78).

The class of chains satisfying Theorem 6 intersects with the classes of chains satisfying the conditions of V. I. Romanovskii ((7), p. 364) and T. A. Sarymsakov ((8), pp. 59, 61, 166). Lemma 2 generalizes a result of Yu. V. Prokhorov (9, 10).

Faculty of Mathematics and Physics, Parhon University
Bucharest, Romania

Received
28 XI 1960

REFERENCES

1. R. L. Dobrushin, *Theory of Probability and Its Applications*, **1**, no. 1, 72 (1956).

2. R. L. Dobrushin, *Theory of Probability and Its Applications*, **1**, no. 4, 365 (1956).

3. A. Kolmogoroff, *C. R.*, **191**, 910 (1930).

4. A. Kolmogoroff, *Math. Ann.*, **99**, 309 (1928).

5. A. N. Kolmogorov, *Basic Concepts of Probability Theory*, Moscow–Leningrad, 1936.

6. B. V. Gnedenko, *A Course in Probability Theory*, Moscow, 1954.

7. V. I. Romanovskii, *Discrete Markov Chains*, Moscow–Leningrad, 1949.

8. T. A. Sarymsakov, *Foundations of the Theory of Markov Processes*, Moscow, 1954.

9. Yu. V. Prokhorov, *Dokl. AN*, **69**, no. 5, 607 (1949).

10. Yu. V. Prokhorov, *Izv. AN SSSR, Ser. Matem.*, **14**, 523 (1950).

Note: Figure translations are in progress. See original paper for figures.