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Abstract

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ON S. L. SOBOLEV' S EMBEDDING THEOREMS FOR ABSTRACT FUNCTIONS

(Presented by Academician S. L. Sobolev on 21 VI 1961)

S. L. Sobolev ([1]) introduced the classes $\Psi_p^{(l)}$ of abstract functions and obtained embedding theorems for these classes.

In the present paper we prove, for the classes $\Psi_p^{(l)}$, an embedding theorem in the case of the limiting exponent ([3]) and a theorem on composite functions. These theorems answer questions posed by S. L. Sobolev ([1]). In the present paper we also study properties of boundary values of functions from $\Psi_p^{(l)}$. For this purpose the classes $\Psi_p^{(\lambda)}$ with arbitrary nonintegral $\lambda \geq 0$ are defined. For boundary values of functions from $\Psi_p^{(1)}$ a theorem is proved that generalizes Gagliardo' s theorem ([4]).

Let D be a bounded domain in the n-dimensional Euclidean space E^n, and let $\Phi(E)$ be an abstract additive function of Lebesgue-measurable subsets of the set D, with values in the Banach space X. The space conjugate to X will be denoted by $\overline{X}$.

Lemma 1. If $\Phi(E) \in \Psi_p(D)$, $\Phi(E) = \int_E \varphi(x)\,dx$, where $\varphi(x)$ is a continuous abstract function with values in X, then

$$\|\Phi(E)\|_{\Psi_p(D)} = \sup_{\|f\|=1} \left[\int_D |f\varphi(x)|^p dx \right]^{1/p}, \tag{1}$$

where $f \in \overline{X}^*$.

Lemma 2. If $\varphi(x)$ is a continuous abstract function of the point $x \in D$ with values in X, then

$$\max_{x\in D} \|\varphi(x)\|_X = \sup_{\|f\|=1} \max_{x\in D} |f\varphi(x)|, \tag{2}$$

where $f \in \overline{X}$.

Lemma 1 is used essentially in the proof of the following theorem.

Theorem 1. Let $lp < n,\ s > n - lp$. Then the function $\Phi(E)$ from $\Psi_p^{(l)}(D)$, on every flat section of the domain D, belongs to $\Psi_q(D_s)$, where

$$q = \frac{sp}{n - lp}.$$

The embedding operator is bounded.**

Let $\Phi(E) = \int_E \varphi(x)\,dx$, where $\varphi(x)$ is an abstract function continuous together with derivatives up to order l. By the trace of the function $\Phi(E)$ on D_s we shall mean the function $\overline{\Phi}(I) = \int_{I \subset D_s} \overline{\varphi}(x)\,dx$, where $\overline{\varphi}(x) = \varphi(x)|_{D_s}$. From Lemma 1 it follows that, for any fixed functional $f,\ f \in X,\ \|f\| = 1$, the real function $f\varphi(x) \in W_p^{(l)}(D)$.

* In the proof of the lemma, properties of the Gelfand integral ([7]) are used.

** For $q < \dfrac{sp}{n - lp}$ the theorem was proved by S. L. Sobolev ([1]).

Applying to the function $f\varphi(x)$ the embedding theorem for the limiting exponent ([2,3]), we obtain

$$\left[\int_{D_s} |f\varphi(x)|^q\,dx\right]^{1/q} \le C_1 \left\{ \left[\int_D |f\varphi(x)|^p\,dx\right]^{1/p} + \sum_{l_1 + \dots + l_n = l} \left[\int_D |fD^l\varphi(x)|^p\,dx\right]^{1/p} \right\}. \tag{3}$$

C_1 does not depend on the function $f\varphi(x)$.

On the basis of Lemma 1 we have

$$\|\widetilde{\Phi}(I)\|_{\Psi_q(D_s)} \le C_1 \|\Phi(E)\|_{\Psi_p^{(l)}(D)}. \tag{4}$$

Inequality (4) makes it possible to define the trace on D_s of any function from $\Psi_p^{(l)}(D)$, using the density in $\Psi_p^{(l)}(D)$ of the set of mean functions and the completeness of $\Psi_q(D_s)$.

We pass to the theorem on composite functions. Define the domains $D, D_y, \Omega, \Omega(t), D_1$ exactly as was done in ([2]), p. 228. Consider a bounded abstract function $\varphi(t, x_1, \dots, x_n; y_1, \dots, y_m)$, defined in the domain D with values in X. We shall assume the function φ to be continuous together with its derivatives with respect to $y_1, \dots, y_m$ up to order l inclusive. For the functions

$$\varphi_{\beta_1, \dots, \beta_m} = \frac{\partial^\beta \varphi}{\partial y_1^{\beta_1} \cdots \partial y_m^{\beta_m}} \qquad (0 \le \beta_1 + \dots + \beta_m = \beta \le l)$$

we shall assume the existence of generalized derivatives up to order l inclusive with respect to $t, x_1, \dots, x_n$ for each fixed system of values $y_1, \dots, y_m$.

Definition. The function $\varphi(t, x_1, \dots, x_n; y_1, \dots, y_m)$ **has property** T (see ([2]), p. 228) if there is a $p > 1$, $p > n/l$, such that for any system of functions $y_i = \eta_i(t, x_1, \dots, x_n)$ the composite functions

$$\left[\varphi^{\alpha_0,\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_m}(t, x_1, \dots, x_n)\right]_{y_i=\eta_i} = \frac{\partial^\alpha \varphi(t, x_1, \dots, x_n, \eta_1(t, x_1, \dots, x_n), \eta_m(t, x_1, \dots, x_n))}{\partial t^{\alpha_0} \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \tag{5}$$

for each fixed t belong to:

$$\Psi_{\frac{1}{\frac{1}{p}-\frac{l-\alpha}{n}}}(\Omega(t)), \qquad \text{if } l \geq \alpha > l - \frac{n}{p},$$

$$K, \qquad \text{if } \alpha < l - \frac{n}{p},$$

where K is the space of abstract continuous functions with the uniform metric; moreover it is assumed that the functions (5) have in the indicated spaces a bound A_Ω (see ([2]), p. 231) independent of α, β and of the system of functions from D_y.

If the functions y_i also depend on $z_1, \dots, z_k$:

$$y_i = \eta_i(t, x_1, \dots, x_n, z_1, \dots, z_k), \tag{6}$$

and, for $z_1, \dots, z_k$ from D_z, the values y_i lie in D_y, then the result of substituting the functions (6) into the function $\varphi(t, x_1, \dots, x_n; y_1, \dots, y_m)$ is a composite function of $t, x_1, \dots, x_n, z_1, \dots, z_k$:

$$\psi(t, x_1, \dots, x_n, z_1, \dots, z_k) = \varphi(t, x_1, \dots, x_n, \eta_1, \dots, \eta_m).$$

Theorem 2. *If the functions $\varphi, \eta_1, \dots, \eta_m$ have property T, then the function ψ also has property T.*

We shall prove the theorem under the assumption that the functions $\varphi_{\beta_1,\dots,\beta_m}$ have strong derivatives with respect to $t, x_1, \dots, x_n$. For any fixed functional $f \in \bar{X}$, $\|f\| = 1$, the function $f\varphi(t, x_1, \dots, x_n, y_1, \dots, y_m)$ has property T with a boundary A_2 independent of f. Application of the lemmas completes the proof of this case. The general case is reduced to the one just considered.

Let S^n be the unit cube of n-dimensional Euclidean space:

$$S^n \equiv \{0 \leq x_i \leq 1,\ i = 1, 2, \dots, n\}.$$

Let $\lambda = l + \alpha > 0$, where l is an integer, $0 < \alpha < 1$. By $\Psi_p^{(\lambda)}(S^n)$ we denote the totality of all abstract additive functions of sets from $\Psi_p^{(l)}(S^n)$ with values in X satisfying the inequality

$$\|\Phi(E)\|_{\Psi_p^{(\lambda)}(S^n)} = \|\Phi(E)\|_{\Psi_p^{(l)}(S^n)} + \sum_{i=1}^{n} \sup_{\|f_i\|=1} \left\{ \int_0^1 t^{-1-p\alpha} \left[\sup_{\tilde{\omega}(x)} \frac{\left| \int_{S_i^t} \tilde{\omega}(x)\, d_x(f\Phi(E+\bar{x}_i^0 t) - f\Phi(E)) \right|}{\|\tilde{\omega}(x)\|_{L_{p'}(S_i^t)}} \right]^p dt \right\}^{1/p} < +\infty, \tag{7}$$

where

$$S_i^t \equiv \begin{cases} 0 \le x_j \le 1, & j = 1, 2, \dots, n,\ j \ne i, \\ 0 \le x_j \le 1 - t, & j = i \end{cases};$$

$\tilde{\omega}(x)$ denotes a real step function taking only a finite number of nonzero values; $E + \bar{x}_i^0 t$ denotes the set obtained by shifting the set E by t in the direction of the x_i-axis; $1/p + 1/p' = 1$.

It is easy to see that when $X = E^1$, $\Psi_p^{(\lambda)}$ is the space of absolutely continuous real functions of sets, isomorphic and isometric to the space $W_p^{(\lambda)}(S^n)$ of functions of n variables. Such spaces were considered by L. N. Slobodetskii ([5]) and by other authors.

The space $\Psi_p^{(\lambda)}(S^n)$ is a Banach space with norm (7). The mean functions $\Phi_h(E)$ form a dense set in $\Psi_p^{(\lambda)}(S^n)$. Moreover, the inequalities

$$\|\Phi_h(E)\|_{\Psi_p^{(l)}(D)} \le C_2 \|\Phi(E)\|_{\Psi_p^{(l)}(D)} \qquad (l = 0, 1, 2, \dots),$$

$$\|\Phi_h(E)\|_{\Psi_p^{(\lambda)}(D)} \le C_3 \|\Phi(E)\|_{\Psi_p^{(\lambda)}(D)}, \qquad \lambda > 0 \text{ nonintegral},$$

hold; C_2 and C_3 do not depend on $\Phi(E)$ and h.

Theorem 3. Let $\Phi(E) \in \Psi_p^{(l)}(S^n)$, where $p > 1$, l is a natural number. Then the traces on

$$S^{n-1} \equiv \{0 \le x_i \le 1,\ i = 1, 2, \dots, n-1;\ x_n = 0\}$$

of the generalized n-derivatives

$$\partial^k \Phi(E)/\partial x_n^k, \qquad k = 0, 1, \dots, l-1,$$

exist in the sense of the definition given above (see the proof of Theorem 1), and belong to the spaces

$$\Psi_p^{(l-k-1/p)}(S^{n-1}).$$

Moreover,

$$\left\| \left. \frac{\partial^k \Phi(E)}{\partial x_n^k} \right|_{x_n=0} \right\|_{\Psi_p^{(l-k-1/p)}(S^{n-1})} \le C_4 \|\Phi(E)\|_{\Psi_p^{(l)}(S^n)},$$

where C_4 does not depend on $\Phi(E)$.

It suffices to prove the theorem for $l = 1$. But in this case application of Lemma 1 reduces the proof to Gagliardo' s theorem (^{4}).

Theorem 4. Let $\widetilde{\Phi}(I) \in \Psi_p^{(1-1/p)}(S^{n-1})$. Then there exists a function $\Phi(E) \in \Psi_p^{(1)}(S^n)$ such that its trace on S^{n-1} is the function $\Phi(I)$.

and the inequality holds

$$\|\Phi(E)\|_{\Psi_p^{(1)}(S^n)} \le C_5 \|\widetilde{\Phi}(I)\|_{\Psi_p^{(1-1/p)}(S^{n-1})};$$

C_5 does not depend on $\widetilde{\Phi}(I)$.

For the mean functions

$$\Phi_h(I) = \int_{I \subset S^{n-1}} \widetilde{\varphi}_h(x)\, dx$$

the desired extensions are provided by the functions

$$\Phi_h(E) = \int_E \frac{1}{x_n^{n-1}} \int_{x_1}^{x_1+x_n} d\xi_1 \dots \int_{x_{n-1}}^{x_{n-1}+x_n} \varphi_h(\xi_1, \dots, \xi_{n-1})\, d\xi_{n-1}\, dx_1, \dots, dx_n.$$

For $p = 1$ the following holds.

Theorem 5. The trace of a function $\Phi(E)$ from $\Psi_1^{(1)}(S^n)$ on S^{n-1} is a function $\widetilde{\Phi}(I)$ from $\Psi_1(S^{n-1})$, and

$$\|\widetilde{\Phi}(I)\|_{\Psi_1(S^{n-1})} \le C_6 \|\Phi(E)\|_{\Psi_1^{(1)}(S^n)}. \tag{8}$$

Conversely, let $\widetilde{\Phi}^1(I) \in \Psi_1(S^{n-1})$. One can construct a function $\Phi(E) \in \Psi_1^{(1)}(S^n)$ such that its trace on S^{n-1} is the function $\widetilde{\Phi}(I)$, and moreover the inequality

$$\|\Phi(E)\|_{\Psi_1^{(1)}(S^n)} \le C_7 \|\widetilde{\Phi}(I)\|_{\Psi_1(S^{n-1})} \tag{9}$$

is satisfied;

C_6, C_7 do not depend on $\widetilde{\Phi}(I)$.

Here too the extension begins with mean functions. In this case the extension operator is nonlinear. In obtaining inequalities (8) and (9), the equivalence of the norm $\| \, \|_{\Psi_1}$ with the uniform norm (1) is essentially used.

Remark 1. The assertions of Theorems 3 and 4 are also valid for the cube S^m, where m, in the case of Theorem 3, satisfies the inequality $l-k-\frac{n-m}{p}>0$, $k=0,1,2,\ldots,l-1$, and, in the case of Theorem 4, the inequality $1-\frac{n-m}{p}>0$.

Remark 2. Using Bochner' s norm ([6]), one can introduce the spaces $B_p^{(l)}(S^n)$, with l a natural number, and $B_p^{(\lambda)}(S^n)$ with nonintegral $\lambda \geq 0$. For such spaces, Theorems 1-5 are formulated and proved analogously.

Remark 3. Lemmas 1 and 2 make it possible to give a new proof of the embedding theorems obtained by S. L. Sobolev with the aid of theorems on integrals of potential type ([1]).

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Note: Figure translations are in progress. See original paper for figures.