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Abstract

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ON THE GEOMETRIC THEORY OF THE SIMPLEST SINGULAR VARIATIONAL PROBLEM FOR AN $(n-1)$-FOLD INTEGRAL

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Let us consider the variational problem for an $(n-1)$-fold integral in parametric form:

$$I = \int_{Q(t^a)} \cdots \int \Phi\left(\xi^\lambda, \xi^\lambda_a\right) dt^1 \ldots dt^{n-1} \quad \left(\xi^\lambda_a = \frac{\partial \xi^\lambda}{\partial t^a}\right). \tag{1}$$

The requirement that the value of the integral (1) be invariant with respect to a transformation of the parameters,

$$^*t^a = C^a(t^b) \quad \left(\operatorname{Det}|C^a_b| > 0,\ C^a_b = \frac{\partial C^a}{\partial t^b}\right), \tag{2}$$

imposes a restriction on the integrand Φ:

$$\Phi\left(\xi^\alpha, \xi^\alpha_b C^b_a\right) = \operatorname{Det}|C^b_a|\, \Phi\left(\xi^\alpha, \xi^\alpha_c\right). \tag{3}$$

It is easy to show that a function Φ satisfying condition (3) has the form

$$\Phi\left(\xi^\alpha, \xi^\lambda_a\right) = \mathfrak{H}\left(\xi^\lambda, \varepsilon_{\alpha\beta_1\ldots\beta_{n-1}} \xi^{\beta_1}_1 \ldots \xi^{\beta_{n-1}}_{n-1}\right), \tag{4}$$

where the function $\mathfrak{H}$ is positive and positively homogeneous of the first degree with respect to the second group of arguments, and $\varepsilon_{\alpha\beta_1\ldots\beta_{n-1}}$ is the unit fundamental n-vector density of weight -1. For the function (4), the integral (1) has the following form:

$$\sigma = \int_{Q(t^a)} \cdots \int \mathfrak{H}\left(\xi^\lambda, \eta_\alpha\right) dt^1 \ldots dt^{n-1} \quad \left(\eta_\alpha = \varepsilon_{\alpha\beta_1\ldots\beta_{n-1}} \xi^{\beta_1}_1 \ldots \xi^{\beta_{n-1}}_{n-1}\right). \tag{5}$$

Interpreting the variables ξ^α as coordinates of points in an n-dimensional geometric space X_n, we shall call every hypersurface of the form

$$\xi^\alpha = \xi^\alpha(t^a) \tag{6}$$

measurable if along it the integral (5) is meaningful. The value of the integral (5) for the hypersurface (6) is called its hyperarea[1].

The concept of the indicatrix of the variational problem (5) is introduced essentially in the same way as for a variational problem expressed by an ordinary integral[2]. For fixed ξ^λ, consider the equation

$$\mathfrak{H}\,(\xi^\lambda, \eta_\alpha) = 1, \tag{7}$$

which defines a hypersurface in the hyperplane coordinates x in the space $\mathfrak{E}_n$ of all contravariant vector densities of weight $+1$.

The variational problem (5) is called a **singular variational problem of singularity class** r, if the rank of the matrix

$$\left\|\mathfrak{H}^{\alpha\beta}\right\|, \qquad \left(\mathfrak{H}^{\alpha\beta} = \frac{\partial^2 \mathfrak{H}}{\partial \eta_\alpha \partial \eta_\beta}\right) \tag{8}$$

is equal to $n-1-r$. For $r=0$ the variational problem (5) is called **regular** (^{1}).

In the present note we shall be interested in the case when $r=1$. Thus the indicatrices (7) will be hypersurfaces of the first class of singularity (^{2}) in the composite manifold (^{3}) $\mathfrak{E}_n(X_n)$.

The remarks just made show that in considering the geometric theory of a singular variational problem for $(n-1)$-fold integrals one must know the theory of a singular hypersurface of the 1st class of singularity in $\mathfrak{E}_n$. Since $\mathfrak{E}_n$, in a certain sense (^{1}), is a centro-affine space, one may make use of the centro-affine theory of a singular hypersurface of the 1st class of singularity (^{2}).

Assuming that the tangent hyperplanes of the singular hypersurface (7) do not pass through the center $\mathfrak{E}_n$, its equation in hyperplane coordinates can be represented in parametric form:

$$\eta_\alpha = l_\alpha(\eta^a) \qquad (a, b, \ldots = 1, 2, \ldots, n-2;\ \alpha, \ldots, \omega = 1, 2, \ldots, n). \tag{9}$$

The fundamental differential equations of the hypersurface (9) are written as follows (^{2}):

$$\nabla_a l_\alpha = l_{aa}, \qquad \nabla_b l_{aa} = -h_{ba} l_\alpha - \mathfrak{G}_{ba}\mathfrak{R}_\alpha, \tag{10}$$

$$\nabla_b \mathfrak{R}_\alpha = -\mathfrak{B}^c{}_b l_{\alpha c} - \mathfrak{B}_b l_\alpha, \tag{11}$$

where h_{ba} is a tensor, $\mathfrak{G}_{ba}$, $\mathfrak{B}^c{}_b$, $\mathfrak{B}_a$ are ω-densities of weights $-\dfrac{2}{n-2}$ and $\dfrac{2}{n-2}$. The $n-1$ covariant vectors of the hypersurface l_α, $l_{\alpha a}$ and the ω-density $\mathfrak{R}_\alpha$ of weight $\dfrac{2}{n-2}$ are linearly independent. The covariant derivatives in (10) and (11) are taken with respect to the connection with coefficients

$$G^c_{ba} = \mathfrak{G}^{cd}\mathfrak{n}^\alpha_d \partial_b l_{\alpha a}, \tag{12}$$

where $\mathfrak{n}^\alpha_a$ is an ω-density of weight $-\dfrac{2}{n-2}$, satisfying the equations

$$l_\alpha \mathfrak{n}^\alpha_a = 0, \qquad l_{\alpha a}\mathfrak{n}^\alpha_b = \mathfrak{G}_{ab}, \qquad \mathfrak{R}_\alpha \mathfrak{n}^\alpha_a = 0, \tag{13}$$

and

$$\mathfrak{G}_{ab}\mathfrak{G}^{bd} = \delta^d_a. \tag{14}$$

Suppose that for $n > 3$ the tensor h_{ba} does not degenerate,

$$\mathfrak{h} = \operatorname{Det}|h_{ba}| \neq 0. \tag{15}$$

As is known (^{1}), with each $\mathfrak{E}_n$ one may associate a centro-affine space E_n, and conversely; moreover, if densities in these spaces of covariant, contravariant valencies and weights respectively $p, q, \mathfrak{k}$ and p, q, k are involved, then the equality holds:

$$k = q - p - (n-1)\mathfrak{k}. \tag{16}$$

With the aid of the quantity

$$\mathfrak{A} = \left(\frac{1}{(n-1)!}\mathfrak{E}^{\alpha\beta\alpha_1\ldots\alpha_{n-2}}\mathfrak{E}^{a_1\ldots a_{n-2}} l_\alpha \mathfrak{R}_\beta l_{\alpha_1 a_1}\cdots l_{\alpha_{n-2}a_{n-2}}\right)^2, \tag{17}$$

which is a scalar density of weight 2 in $\mathfrak{E}_n$, and, according to (16), of weight $-2(n-1)$ in E_n and of weight $\dfrac{2n}{n-2}$ on the hypersurface, we shall put in corres
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quantities from $\mathfrak{E}_n$ to the improper quantities from E_n (1) by means of the equalities

$$\tilde{l}_\alpha = |\mathfrak{h}|^{\frac{n}{2(n-1)(n-2)}}\,\mathfrak{A}^{-\frac{1}{2(n-1)}}\, l_\alpha(\eta^a), \tag{18}$$

$$ {}^{*}\tilde{l}_{aa} = |\mathfrak{h}|^{\frac{n}{2(n-1)(n-2)}}\, \mathfrak{A}^{-\frac{1}{2(n-1)}}\, l_{aa}(\eta^{b}), \tag{19} $$

$$ \tilde{N}_{\alpha} = |\mathfrak{h}|^{-\frac{1}{2(n-1)}}\, \mathfrak{A}^{-\frac{1}{2(n-1)}}\, \mathfrak{R}_{\alpha}(\eta^{b}), \tag{20} $$

for which the derivational equations (10) and (11) take the form:

$$ \nabla_{a}\tilde{l}_{\alpha} = {}^{*}\tilde{l}_{aa} - \frac{n}{(n-1)(n-2)} A_{a}\tilde{l}_{\alpha}, \tag{21} $$

$$ \nabla_{b}{}^{*}\tilde{l}_{aa} = -\frac{n}{(n-1)(n-2)} A_{b}{}^{*}\tilde{l}_{aa} - h_{ba}\tilde{l}_{\alpha} - \tilde{g}_{ba}\tilde{N}_{\alpha}, \tag{22} $$

$$ \nabla_{a}\tilde{N}_{\alpha} = -\tilde{V}^{c}_{\cdot a}\, {}^{*}\tilde{l}_{ac} - \tilde{W}_{a}\tilde{l}_{\alpha} + \frac{1}{n-1} A_{a}\tilde{N}_{\alpha}, \tag{23} $$

where

$$ \tilde{g}_{ba} = |\mathfrak{h}|^{-\frac{1}{n-2}}\, \mathfrak{G}_{ba}, \tag{24} $$

$$ \tilde{V}^{c}_{\cdot a} = |\mathfrak{h}|^{-\frac{1}{n-2}}\, \mathfrak{V}^{c}_{\cdot a}, \tag{25} $$

$$ \tilde{W}_{a} = |\mathfrak{h}|^{-\frac{1}{n-2}}\, \mathfrak{W}_{a}, \tag{26} $$

$$ A_{a} = -\frac{1}{2}\nabla_{a} \ln|\mathfrak{h}|. \tag{27} $$

Alongside the tangent composite manifold of the first order $E_n(X_n)$, consider the composite manifold $\mathfrak{E}_n(X_n)$. The specification of the singular variational problem (5) entails the specification of a field of singular hypersurfaces in the composite manifold $\mathfrak{E}_n(X_n)$. Assuming now that, in (9)–(27), all the quantities considered also depend on ξ^{α}, the equations of the field of singular hypersurfaces may be written in the form

$$ \eta_{\alpha} = l_{\alpha}(\xi^{\lambda}, \eta^{a}) \tag{28} $$

in hyperplane coordinates. Considering each hypersurface of the field (28) as X_{n-2}, we arrive at consideration of the composite manifold $X_{n+(n-2)}$. The principal task in studying this composite manifold is to find an invariant linear connection, determined by solving the Pfaff equations

$$d\eta^a + \Gamma^a = 0, \tag{29}$$

where $\Gamma^a_\alpha(\xi^\lambda, \eta^b)\, d\xi^\alpha$ are the required Pfaff forms.

Corresponding to the quantities (18)–(20) and (28), introduce into consideration n independent Pfaff forms:

$$\tilde{l} = \tilde{l}_\alpha(\xi^\lambda, \eta^b)\, d\xi^\alpha, \qquad {}^*\tilde{l}_a = {}^*\tilde{l}_{aa}(\xi^\lambda, \eta^b)\, d\xi^\alpha, \qquad \tilde{N} = \tilde{N}_\alpha(\xi^\lambda, \eta^a)\, d\xi^\alpha. \tag{30}$$

Representing the operator of base differentiation (^{3}) in the form of the symbolic equality

$$D = {}^*\tilde{l}_a D^a + \tilde{l}\, D_{(n-1)} + \tilde{N}\, D_{(n)}, \tag{31}$$

and the required forms in the form of the expansion

$$\Gamma^a = \gamma^{ab*}\tilde{l}_b + \gamma^a \tilde{l} + \delta^a \tilde{N}, \tag{32}$$

one can prove the following assertion.

If the indicatrices of the singular variational problem for an $(n-1)$-fold integral are not cylindrical or conical surfaces, and their asymptotic cones are hypercones, while the tensor h_{ab} is nondegenerate, then the invariant linear connection in the composite manifold determined by the field of indicatrices, for $n > 3$, is uniquely determined from the conditions

$$[\tilde{l}D\tilde{l}] = 0, \qquad [\tilde{N}D\tilde{l}] = 0, \qquad \underset{(n)}{D}\, \tilde{g}_{ab} = 0, \tag{33}$$

where the square brackets denote the exterior product of forms in the sense of Cartan. For $n = 3$ the third condition is replaced by the condition

$$\underset{(N)}{D}\, \mathfrak{W}^{(3)} = 0, \tag{34}$$

where $\mathfrak{W}^{(3)}$ is a scalar density of weight 3, replacing the w-density $\mathfrak{W}_a$.

The linear connection in $X_{n+(n-2)}$ found in this way can be used to find conditions for reducibility of the field of singular hypersurfaces in $\mathfrak{E}_n(X_n)$ to a constant one, i.e. essentially the conditions for reducibility of the variational problem (5) to the case when the integrand contains no variables ξ^a. The conditions found are indicated by the following

Theorem. In order that the field of singular hypersurfaces of the type under consideration in $\mathfrak{E}_n(X_n)$ be reducible, by means of a suitable choice of coordinate systems, to a constant one, it is necessary and sufficient that, for $n > 3$, the linear connection in the composite manifold be a connection of zero curvature, that the fields of local objects G^c_{ab}, $\tilde{g}_{ab}$, and h_{ab} be constant with respect to this connection, and that the scalar φ be identically equal to zero, where φ is the coefficient of the bracket $[\tilde{l}\tilde{N}]$ in the expansion of $[D\tilde{l}]$.

For $n = 3$ the indicated conditions are simplified.

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Note: Figure translations are in progress. See original paper for figures.

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