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Abstract

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ON THE WEIGHT AND CARDINALITY OF H-CLOSED SPACES

(Presented by Academician P. S. Aleksandrov on 10 II 1961)

In (^{1}) P. S. Aleksandrov showed that under a continuous mapping of a bicompact space onto a Hausdorff space the weight does not increase. From the lemma of P. S. Aleksandrov and P. S. Urysohn on the coincidence of the pseudocharacter with the weight at points of a bicompact space $((^3), p.889)$, it further follows easily that the weight of a bicompact space is not greater than its cardinality.

Below (unless the contrary is stipulated) all spaces are assumed to be Hausdorff, and a space is called **noncompactifiable** if every one-to-one continuous mapping of it onto a Hausdorff space is a homeomorphism. A space is called **canonical** if it has a base consisting of canonical open sets, and a T_u**-space** if any two of its distinct points have neighborhoods with disjoint closures. Katětov (^{4}) proved that every H-closed space is compactified (is mapped one-to-one and continuously) onto a noncompactifiable one, that the class of noncompactifiable spaces coincides with the class of H-closed canonical spaces, and that H-closed T_u-spaces are compactified onto a bicompact space. It is also known that bicompacta are a special case both of H-closed and of noncompactifiable spaces.

The aim of the present paper is two theorems extending the above propositions to a class of spaces broader than the bicompacta.

Theorem 1. *Under a continuous mapping φ of an H-closed space X onto a canonical (noncompactifiable) space $\widetilde{X}$, the weight does not increase.*

Theorem 2. *The weight of a noncompactifiable space X cannot be greater than its cardinality.*

Remark 1. All examples previously known to us of spaces whose weight is greater than their cardinality were not H-closed; however, below we shall give an example of an H-closed space X with weight greater than its cardinality. This example will show that Theorem 2 ceases to be true for arbitrary H-closed spaces. In addition, it will also show that under compactifications of an H-closed space the weight may strictly decrease and, in particular, that there exist

H-closed spaces of different weight which are θ-homeomorphic in the sense of Fomin (^{6}) (for our compactification is a θ-homeomorphism).

Proof of Theorem 1. Without loss of generality one may assume that $\widetilde{X}$ is a partition of X into closed sets and that φ is the natural mapping of X onto $\widetilde{X}$. If $\widetilde{M}$ is a set in $\widetilde{X}$, then by M we shall denote $\varphi^{-1}(\widetilde{M}) = \bigcup \widetilde{M}$. First of all, let us note for what follows that, for any open Γ in X,

$$\varphi([\Gamma]) = [\varphi(\Gamma)].$$

Indeed, on the one hand, always $\varphi([M]) \subset [\varphi(M)]$; on the other hand, $[\Gamma]$ is canonical in X and therefore, in view of the H-closedness of X, by Katětov’ s theorem (^{4}), it is H-closed, so that $\varphi([\Gamma])$ is closed in $\widetilde{X}$ (as the image of an H-closed space). Now the required assertion follows from the inclusions

$$\varphi(\Gamma) \subseteq \varphi([\Gamma]) \subseteq [\varphi(\Gamma)].$$

Our theorem, obviously,

will be proved if the following proposition, which is itself of some interest, is proved:

For any open base $\mathfrak{G} = \{\Gamma_\lambda\}$ of an H-closed space X and any continuous mapping φ of this space onto the canonical space $\widetilde{X}$, the system of all possible finite intersections

$$\bigcap_{i=1}^{n} \left(\widetilde{X} \setminus [\varphi(\Gamma_{\lambda_i})]\right)^{*} \qquad (\Gamma_{\lambda_i} \in \mathfrak{G})$$

forms an open base in $\widetilde{X}$.

Let $\tilde{x} \in \widetilde{X}$, and let $\widetilde{O}\tilde{x}$ be an arbitrary canonical neighborhood of $\tilde{x}$ in $\widetilde{X}$; then $\widetilde{X} \setminus \widetilde{O}\tilde{x} = \widetilde{\Phi}$ is a canonical closed set. Let $\widetilde{G}$ be an open set in $\widetilde{X}$ such that $[\widetilde{G}] = \widetilde{\Phi}$. Using the fact that $\widetilde{X}$ is Hausdorff, for each point $\tilde{y} \in \widetilde{\Phi}$ choose a neighborhood $\widetilde{O}\tilde{y}$ such that $\tilde{x} \notin [\widetilde{O}\tilde{y}]$. Then $\widetilde{O}\tilde{y} = \bigcup \widetilde{O}\tilde{y}$ is a neighborhood of the set $\tilde{y}$ in X. For each $\tilde{y}$ choose from the base $\mathfrak{G} = \{\Gamma_\lambda\}$ a collection of such $\Gamma_{\lambda'} = \Gamma_{\lambda'}(\tilde{y})$ that $\widetilde{O}\tilde{y} = \bigcup_{\lambda'} \Gamma_{\lambda'}(\tilde{y})$. Since $\{\widetilde{O}\tilde{y}\}$ is a cover of $\widetilde{\Phi}$ in $\widetilde{X}$, the possible $\Gamma_{\lambda'}(\tilde{y})$: $\{\Gamma_{\lambda'}(\tilde{y})\} = \gamma$ form a cover of Φ and, a fortiori, a cover of $[G]$ in X, for $\varphi([G]) = [\varphi(G)] = [\widetilde{G}] = \widetilde{\Phi}$. Since $[G]$ is canonical in the H-closed space X, it is H-closed, and in γ there exists a finite subsystem $\{\Gamma_{\lambda_i}\}$ $(i = 1, 2, \dots, n)$ such that $\bigcup_{i=1}^{n} [\Gamma_{\lambda_i}] \supseteq [G]$. Therefore

$$\bigcap_{i=1}^{n} \left(\widetilde{X} \setminus [\varphi(\Gamma_{\lambda_i})]\right) = X \setminus \varphi\left(\bigcup_{i=1}^{n} [\Gamma_{\lambda_i}]\right) \subseteq \widetilde{X} \setminus \varphi([G]) =$$

$$= \widetilde{X} \setminus [\varphi(G)] = \widetilde{X} \setminus [\widetilde{G}] = \widetilde{O}\tilde{x}.$$

It remains to prove that

$$\tilde{x} \in \bigcap_{i=1}^{n} \left(\widetilde{X} \setminus [\varphi(\Gamma_{\lambda_i})]\right)$$

or that $\tilde{x} \bar{\in} \varphi([\Gamma_{\lambda_i}])$ $(i = 1, 2, \ldots, n)$. But $\Gamma_{\lambda_i} = \Gamma_{\lambda'}(\tilde{y})$ for some $\tilde{y} \in \widetilde{\Phi}$, and $\Gamma_{\lambda'}(\tilde{y}) \subseteq \widetilde{O}\tilde{y}$, whence $\varphi(\Gamma_{\lambda'}(\tilde{y})) \subseteq \widetilde{O}\tilde{y}$. Therefore

$$\varphi([\Gamma_{\lambda_i}]) = \varphi([\Gamma_{\lambda'}(\tilde{y})]) = [\varphi(\Gamma_{\lambda'}(\tilde{y}))] \subseteq [\widetilde{O}\tilde{y}],$$

and the last set does not contain $\tilde{x}$ by construction. Theorem 1 is proved.

Let x be a point of the space X (not necessarily Hausdorff). We shall call a system Σ of neighborhoods of x an H-pseudobase of X at x if the intersection of the system of closures of all elements of Σ consists of the point x; and the least of the cardinalities of the H-pseudobases of X at x (if such exist) will be called the H-pseudoweight of X at x and denoted by $\psi_H(x)$.

A T_1-space may be defined as a space in which every point has a pseudobase; analogously, a T_2-space may be defined as one in which every point has an H-pseudobase. It is obvious that in a T_2-space one always has $\psi(x) \leq \psi_H(x) \leq \psi(x)$, where $\psi(x)$ is the pseudoweight and $\psi(x)$ is the weight at x, but in a T_3-space $\psi(x) = \psi_H(x)$, so that Lemma 1 is a strengthening of the above-mentioned proposition of P. S. Aleksandrov and P. S. Uryson.

Lemma 1. *At every point of a noncompact space the H-pseudoweight is equal to the weight.*

Let $\Sigma = \{U(x)\}$ be an H-pseudobase at x, and let us prove that the collection of all finite intersections of elements of Σ forms a base at x.

Let Ox be an arbitrary neighborhood of x; take a canonical neighborhood O_0x such that $O_0x \subseteq Ox$. But

$$\bigcap_{U \in \Sigma} [U(x)] = \{x\},$$

so that

$$(X \setminus O_0x) \cap \bigcap_{u \in \Sigma} [U(x)] = \Lambda,$$

and since, being noncompact, X is H-closed, in the system of closed sets consisting of the closures of all elements of Σ and the set $X \setminus O_0x$, there exists a finite subsystem

$$X \setminus O_0 x, \quad [U_1(x)],\ [U_2(x)], \dots,\ [U_n(x)],$$

the intersection of whose open kernels is empty (see, for example, ([5], p. 130, (AF')). Consequently,

$$\bigcap_{i=1}^{n} U_i(x) \subseteq [O_0 x],$$

and since $O_0 x$ is canonical, it follows that

$$\bigcap_{i=1}^{n} U_i(x) \subseteq O_0 x \subseteq Ox,$$

from which the lemma follows.

Proof of Theorem 2. Obviously, it is enough to show that at each point $x \in X$ there exists an H-pseudobase of cardinality no greater than the cardinality of X. For this purpose, to each pair of points (x, y), $x \neq y$, we assign disjoint neighborhoods $U_y(x), U(y)$. Since $\bar{y} \in [U_y(x)]$, the family $\{U_y(x)\}$ is the required H-pseudobase.

Corollary 1. Every canonical space X has a canonical H-closed (noncompact) extension hX of the same weight, and also a canonical H-closed (noncompact) extension σX, maximal in Fomin' s sense.

For the proof it is enough first to take H-closed extensions with the required property and then compactify them to canonical ones. In doing so the topology of X does not change, since X is canonical.

Corollary 2. An H-closed T_u-space with a countable base is compactified to a compactum.

This follows from the fact that, according to Katětov' s theorem, an H-closed T_u-space is compactified to a bicompactum.

Example. As the points of X we take all ordinal numbers $\leqslant \omega^2$. As neighborhoods of any $\xi < \omega^2$ we take the usual intervals containing ξ, and

$$O\omega^2 = (\xi, \omega^2] \setminus \{\omega \cdot n_k\},$$

where $\{n_k\}$ is an arbitrary rapidly increasing sequence of Urysohn natural numbers (([3]), p. 206):

$$\lim_{k \to +\infty} \frac{n_k}{k} = +\infty.$$

This space is countable, H-closed, and of uncountable weight.

Remark 2. In connection with what has been set forth above, there arises the natural problem of extending to noncompact spaces the addition theorem of Smirnov—Arhangel' skii for the weight of sets in bicompacta (2) (our example shows that such an addition theorem is not true for the weight of sets in arbitrary H-closed spaces). In this connection we know only the following fact:

If the space X is canonical and M is everywhere dense in X, then M has in X an outer base of cardinality equal to the weight of the space M.

This is easy to prove if one uses the operator $O(\Gamma)$.

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Note: Figure translations are in progress. See original paper for figures.