ON ITO’ S STOCHASTIC INTEGRAL EQUATION

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Abstract

Full Text

I. V. GIRSANOV

ON ITO' S STOCHASTIC INTEGRAL EQUATION

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1. In the present note we study Ito' s stochastic equation

$$x_t(\omega) = x_s(\omega) + \int_s^t \sigma(u, x_u(\omega))\, d\xi_u(u) + \int_s^t m(u, x_u(\omega))\, du, \quad s \le u \le t \le T. \tag{1}$$

Here $\sigma(u,x) = \|\sigma^i_j(u,x)\|$ is a matrix; $m(u,x) = \{m^i(u,x)\}$ is a vector of the n-dimensional Euclidean space R^n; $x_t(\omega) = \{x^i_t(\omega)\}$ is a stochastic process in R^n continuous in t; $\xi_t(\omega) = \{\xi^i_t(\omega)\}$ is an n-dimensional Wiener process. By $\mathbf{P}(\cdot)$ we shall denote the probability distribution in the space Ω of elementary outcomes; by $\mathbf{E}(\cdot \mid A)$, conditional mathematical expectation with respect to the measure $\mathbf{P}$ under the condition A; by μ_x, the measure in the space C of continuous functions of t, $s \le t \le T$, with values in R^n, generated by the process $x_t(\omega)$ (in what follows ω will, as a rule, be omitted).

Ito (1) and Gikhman (2) proved the existence and uniqueness of the solution of equation (1), assuming that the coefficients $\sigma(u,x)$ and $m(u,x)$ satisfy a Lipschitz condition in x and the inequality

$$\sum_{i,j} |\sigma^i_j(u,x)| + \sum_i |m^i(u,x)| < c\left(1 + \sum_i |x^i|\right).$$

It can be shown (see, for example, (1)) that fulfillment of this inequality makes it possible to reduce the study of equation (1) to the case of bounded coefficients.

2. **Existence theorems.** A. V. Skorokhod obtained in (3) an existence theorem for (1) in the case of continuous coefficients. Additional restrictions on the diffusion coefficients of the process in a neighborhood of the discontinuity points make it possible to strengthen this result.

Lemma 1. Let η_t be a random function continuous in t, connected with a one-dimensional Wiener process ξ_t by the relation

$$\eta_t = \eta_s + \int_s^t \varphi(u,\omega)\, d\xi_u + \int_s^t \psi(u,\omega)\, du.$$

Let $G_0 = \{x : \ |x-a| < 2\varepsilon\}$, $G_1 = \{x : \ |x-a| < \varepsilon\}$. Suppose that, when $\eta_t \in G_0$, almost surely

$$0 < c_1 \leq \varphi(u,\omega)^2 \leq c_2, \qquad |\psi(u,\omega)| < c_3.$$

Then for any $\Gamma \in G_1$ the inequality

$$\mathbf{P}\{\mu_1(u : \ \eta_u \in \Gamma, \ s \leq u \leq T) > \delta\} \leq (c_1\delta)^{-1/2} C(c_2, c_3, \varepsilon)\, \mu_1(\Gamma)^{1/2},$$

holds, where μ_1 is the one-dimensional Lebesgue measure.

The proof of Lemma 1 is based on the following fact:

Lemma 2. Let

$$\eta_t = \int_s^t \varphi(u,\omega)\, d\xi_u.$$

Let

$$t = s + \int_s^{\tau(\omega)} \varphi^2(u,\omega)\, du.$$

Let $0 < c_1 \leq \varphi^2(u,\omega) \leq c_2$. Then the process $x_t(\omega) = \eta_{\tau_t(\omega)}(\omega)$ is a Wiener process*.

We shall call a set γ in $R^n \times [s,T]$ **admissible** for $\sigma(u,x)$ if it is of type G_δ and, in some coordinate system, is contained in the plane $\{x^1 = 0\}$, while its projection onto the plane (t, x^1) has measure zero, and

$$\sum_i \sigma_i^1(t,x)^2 \geq \sigma_0 > 0 \quad \text{for } |x^1| < \varepsilon. \tag{2}$$

An example of such a γ may be a smooth $(n-1)$-dimensional manifold in R^n.

Theorem 1. *Let the coefficients of the matrix $\sigma(u,x)$ and of the vector $m(u,x)$ be bounded and continuous everywhere except for a certain number of admissible sets γ. Then there exists a process $x_t(\omega)$ that is a solution of equation (1).*

The proof of the theorem is carried out according to the following scheme. By virtue of the restrictions on γ, one can construct sequences $\sigma_n(u,x)$ and $m_n(u,x)$ satisfying, for each n, the Lipschitz condition and converging to $\sigma(u,x)$ and $m(u,x)$ uniformly everywhere except for any neighborhood of γ. Since condition (2) may fail to hold for $\sigma_n(u,x)$, we adjoin to ξ_t an independent Wiener process η_t and consider the sequence of processes satisfying the equations ($\alpha > 0$, $n = 1, 2, \ldots$)

$$x_t(n,\alpha) = x_s(n,\alpha) + \int_s^t \sigma_n(u, x_u(n,\alpha))\, d\xi_u + \alpha\eta_t + \int_s^t m_n(u, x(n,\alpha))\, du. \quad (3)$$

Since $E\left(\sum_i [x^i_{t+h}(n,\alpha) - x^i_t(n,\alpha)]^4\right) \leq ch^2$, the family $\mu_{x(n,\alpha)}$ is weakly compact in C. We may assume that, as $n \to \infty$, $\mu_{x(n,\alpha)}$ converges weakly to some distribution μ^α. For a suitable choice of Ω, relying on the results of A. V. Skorokhod (^{5}), §3, one can construct sequences $\tilde{x}_t(n,\alpha)$, $\tilde{\xi}_t(n,\alpha)$, $\tilde{\eta}_t(n,\alpha)$ of processes in C, having the same joint probability distribution as $x_t(n,\alpha)$, $\xi_t(n,\alpha)$, $\eta_t(n,\alpha)$, and converging, with probability 1, to limiting processes $\tilde{x}_t(\alpha)$, $\tilde{\xi}_t(\alpha)$, $\tilde{\eta}_t(\alpha)$.

The processes $\tilde{x}_t(n,\alpha)$, $\tilde{\xi}_t(n,\alpha)$, and $\tilde{\eta}_t(n,\alpha)$ are connected with one another by equations (3), in which one can pass to the limit as $n \to \infty$, since from the properties of stochastic integrals and the estimate of Lemma 1 (with $c_1 = \alpha^2$) it follows that the integrals on the right-hand side of (3) converge in probability to the corresponding integrals of the limits.

Carrying out this limiting transition, we obtain for $\tilde{x}_t(\alpha)$ the equation

$$\tilde{x}_t(\alpha) = \tilde{x}_s(\alpha) + \int_s^t \sigma(u, \tilde{x}_u(\alpha))\, d\tilde{\xi}_u(\alpha) + \alpha\tilde{\eta}_t(\alpha) + \int_s^t m(u, \tilde{x}_u(\alpha))\, du. \quad (4)$$

For the solution of equation (4), in the estimate of Lemma 1 $c_1 = \alpha^2 + \sigma_0^2$; this makes it possible to carry out in (4) a repeated limiting passage as $\alpha \to 0$ and thereby prove the theorem.

The nondegeneracy of $\sigma(u,x)$ makes it possible to substantially weaken the restrictions on the vector $m(u,x)$. From the results of (^{6}) it follows:

Theorem 2. *Suppose equation (1) has a solution x_t. Then the equation*

$$\tilde{x}_t = \tilde{x}_s + \int_s^t \sigma(u, \tilde{x}_u)\, d\tilde{\xi}_u + \int_s^t [m(u, \tilde{x}_u) + \sigma(u, \tilde{x}_u)\widetilde{m}(u, \tilde{x}_u)]\, du, \quad (5)$$

also has a solution, where $\widetilde{m}(u,x)$ is a vector with bounded coordinates.

* An analogous transformation was used in the theory of Markov processes by V. A. Volkonskii (^{4}).

For the proof one must carry out the transformation described in (6),

$$\widetilde{\mathbf{P}}(d\omega) = \exp[\zeta(\widetilde{m})]\, \mathbf{P}(d\omega),$$

taking as the initial Itô process the process x_t.

3. Uniqueness theorems. In (3) the uniqueness theorem was proved for $n = 1$ under the assumption that the coefficients satisfy Hölder conditions with exponent $\alpha > 1/2$ and $\sigma \geq \sigma_0 > 0$. In the present work the question of uniqueness of the solution of the equation is reduced to the question of the existence of a sufficiently smooth solution of the parabolic differential equation:

$$\frac{\partial v(s,x,t)}{\partial t} = \sum_{i,j} a^{ij}(t,x) \frac{\partial^2 v}{\partial x^i \partial x^j} + m^i(t,x) \frac{\partial u}{\partial x_i}, \tag{6}$$

where $v(s,x,s) = f(x)$,

$$a^{ij}(t,x) = \frac{1}{2} \sum_k \sigma_k^i(t,x) \sigma_k^j(t,x), \qquad t \geq s.$$

Lemma 3. Let $x_t(\omega)$ and $y_t(\omega)$ be two solutions of equation (1), with $\mu_x = \mu_y = \mu$, $\xi_t(\omega) = \xi_t(\tilde{\omega})$ for $s \leq t \leq T$ only when $\omega = \tilde{\omega}$. Then, if the matrix $\sigma(t,x)$ is nondegenerate, $x_t(\omega) = y_t(\omega)$ for $s \leq t \leq T$.

For the proof we note that, by virtue of the nondegeneracy of $\sigma(t,x)$, in C there exists almost everywhere with respect to the measure μ the transformation

$$F(z(\cdot))_t = \int_s^t \sigma^{-1}(u, z_u(\cdot)) \left[dz_u(\cdot) - m(u, z_u(\cdot))\, du \right].$$

A direct calculation shows that $F(x(\omega)) = F(y(\omega)) = \xi_t(\omega)$. It follows that if $x_t(\omega) = y_t(\tilde{\omega})$, $t \in [s,T]$, then $\xi_t(\omega) = \xi_t(\tilde{\omega})$ for $t \in [s,T]$, and hence $\omega = \tilde{\omega}$. Put $\omega \in A$ if, for some $\tilde{\omega}$ and all $t \in [s,T]$, $x_t(\omega) = y_t(\tilde{\omega})$; the assertion of the lemma follows from the fact that, by virtue of the coincidence of μ_x and μ_y, $\mathbf{P}(A) = 1$.

We shall call a set of bounded continuous functions f on R^n **dense** if it is everywhere dense in the space of bounded continuous functions tending to zero as $x \to \infty$. From the maximum principle for equation (6) and the Riesz theorem on the general form of a linear continuous functional it follows that

$$v(s,x,t) = \int f(y)\, \pi(s,x,t,dy), \tag{7}$$

where $\pi(s,x,t,\Gamma)$ is a probability measure in R^n.

Theorem 3. Let $\sigma(u,x)$ and $m(u,x)$ be such that equation (6), for each function f from a dense set, has a solution whose second derivatives with respect to x are continuous everywhere except for some set γ admissible for σ. If x_t is a solution of equation (1), then x_t is a Markov process with transition function (7) $\pi(s,x,t,\Gamma)$. If the matrix $\sigma(t,x)$ is nondegenerate, then this solution is unique.

For the proof we note that from Itô' s formula (8) for the change of variables in the stochastic integral (the legality of applying it in the case of discontinuity of the second derivatives of v follows from Lemma 1) one can derive the equality

$$\mathbf{E}\left[f(x_t) \mid x_u,\ s \le u \le \sigma\right] = v(\sigma, x_\sigma, t).$$

By virtue of (7) it follows that

$$\mathbf{P}\{x_t \in \Gamma \mid x_u,\ s \le u \le \sigma\} = \pi(\sigma, x_\sigma, t, \Gamma).$$

Thus it follows that (7), x_t is a Markov process with transition function $\pi(s, x, t, \Gamma)$. Since μ_x is determined by x_s and $\pi(s, x, t, \Gamma)$ uniquely, for any other solution y_t, $\mu_x = \mu_y$. From this and Lemma 3 follows the last assertion of the theorem.

The existence of a "good" solution of equation (7) has been proved in the case of $\sigma(u, x)$ and $m(u, x)$ satisfying a Hölder condition with arbitrary α, if

$$\sum_{i,j} a^{ij}(u, x)\, \xi_i \xi_j \ge c^2 \sum_i (\xi_i)^2.$$

The author has succeeded in extending this result to the case of discontinuous $m(u, x)$ and $\sigma(u, x)$, requiring additional smoothness on the surfaces of discontinuity (9).

Theorem 4. *Suppose that for any two solutions x_t and y_t of equation (1), $\mu_x = \mu_y$. Then the same holds for any solutions $\tilde{x}_t$ and $\tilde{y}_t$ of equation (5).*

The proof follows from the results of (6), since

$$\frac{d\mu_{\tilde{x}}}{d\mu_x} = \frac{d\mu_{\tilde{y}}}{d\mu_y}.$$

Finally, let us note that abandoning the requirement of nondegeneracy of the matrix $\sigma(u, x)$ even at one point can lead to an actual violation of uniqueness. An example is furnished by the equation:

$$x_t = x_s + \int_s^t |x_u|^\alpha \, d\xi_u.$$

For $\alpha \ge \frac{1}{2}$ it has a unique solution; for $0 < \alpha < \frac{1}{2}$ uniqueness is absent. The infinitesimal operators of homogeneous Markov processes satisfying this equation coincide everywhere except at the point zero, where they are given by the formulas

$$\begin{aligned} &Af(0) = c^{-1}[f'(+0) - f'(-0)] && \text{for } c \neq 0,\ c \neq \infty; \\ &Af(0) = \lim_{x\to 0} |x|^{2\alpha} f''(x) && \text{for } c = 0; \\ &Af(0) = 0 && \text{for } c = \infty. \end{aligned}$$

4. The solution of stochastic equations whose coefficients have discontinuities on smooth manifolds makes it possible to construct diffusion processes with reflection at the boundary by means of the actual reflection of their trajectories from the boundary of the domain. Further investigation makes it possible, using a generalization of Itô' s formula (8), to introduce into equation (1) a term of the type of an integral of a δ-function, describing the effect of reflection. The results obtained can be used in clarifying conditions for the absolute continuity or singularity of measures associated with diffusion processes with boundary.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.

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