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Abstract

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THREE-DIMENSIONAL RELATIVISTIC SCHRÖDINGER EQUATION FOR THE TWO-BODY PROBLEM

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The aim of the present work is a relativistic generalization of the Schrödinger equation without increasing the number of independent variables; this is the main distinction of the equation obtained here from the well-known Bethe-Salpeter equation ([1]). We shall construct the Schrödinger equation in the velocity representation. By the symbols $v, w, ...$ we shall mean four-velocity vectors. The scalar product of vectors is written in the form $vw = v_0 w_0 - \mathbf{vw}$. All four-velocity vectors satisfy a relation of the form $v^2 = 1$. Velocity space is invariant with respect to Lorentz transformations, i.e., it is a relativistic object. From the geometrical point of view, velocity space may be regarded as a three-dimensional Lobachevsky space, with the Lorentz group being the group of motions of this space ([2]). In other words, Lorentz-invariant quantities composed of velocities are geometrical quantities of Lobachevsky space. In this connection one may assert (somewhat schematically) that the transition from classical mechanics to relativistic mechanics consists in replacing Euclidean geometry by Lobachevsky geometry in velocity space. The existence of this geometrical parallel is very useful in comparing the corresponding relativistic and nonrelativistic concepts. The metric in velocity space is introduced as follows. If the (non-Euclidean) distance between two points v_1 and v_2 is equal to s, then the relations

$$\operatorname{ch} s = v_1 v_2; \tag{1}$$

$$4 \operatorname{sh}^2 \frac{s}{2} = -(v_1 - v_2)^2. \tag{2}$$

hold.

The physical meaning of the non-Euclidean metric is revealed by the relation (2)

$$\operatorname{th} s = v_{\text{rel}}, \tag{3}$$

where v_{rel} is the relative velocity of two particles which, in the given reference frame, have velocities v_1 and v_2.

Let us consider the problem of the interaction of two spinless particles with masses m_1 and m_2. In the chosen representation the system is described by a wave function $\psi(v_1, v_2)$. Instead of the velocities v_1 and v_2, it is convenient to introduce the velocities v and w, defined by the equalities

$$v = \frac{m_1}{M_0} v_1 + \frac{m_2}{M_0} v_2; \tag{4}$$

$$w = -\frac{m_2}{M_0} v_1 + \frac{m_1 + 2m_2 \operatorname{ch} s}{M_0} v_2, \tag{5}$$

where

$$M_0 \equiv M_0(s) = \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \operatorname{ch} s}; \tag{6}$$

$$\operatorname{ch} s = v_1 v_2 = vw. \tag{7}$$

Equality (7) means that the “distance” between v and w is equal to the “distance” between v_1 and v_2. It can be shown that the segment (geodesic line) vw is obtained as a result of a parallel displacement of the segment $v_1 v_2$.

The velocity v, by definition, will be called the velocity of the center of mass of the system. We note that the velocity of the center of mass is defined in a purely kinematic way and does not depend on the character of the interaction between the particles. The velocity w, as will be clear from what follows, plays the role of a relative velocity. To describe the interaction, i.e., the dynamics of the system, an energy-momentum operator is needed, which we define by the equality

$$\hat{P}_\mu = \hat{M} v_\mu. \tag{8}$$

Here v_μ is the velocity of the center of mass (4), and $\hat{M}$ is the operator of the total mass of the system. The operator $\hat{M}$ must satisfy the following requirements: 1) $\hat{M}$ is a relativistic invariant; 2) $\hat{M}$ commutes with the velocity vector of the center of mass. Requirement 2) follows from the fact that in relativistic theory the velocity of the center of mass must be a constant of the motion. Thanks to this requirement the definition (8) becomes noncontradictory, and the velocity of the center of mass (4) acquires physical meaning.

By analogy with the nonrelativistic theory, we shall seek the operator $\hat{M}$ in the form

$$\hat{M} = M_0 + \hat{U},$$

where the total mass of the "free" particles M_0 is given by equality (6). As $\hat{U}$ one may take an integral operator whose kernel depends on the relative velocities. For the concrete construction of the operator $\hat{U}$, it is convenient to represent the vector w in the form

$$w = v \operatorname{ch} s + z \operatorname{sh} s, \tag{9}$$

where the vector z satisfies the relations

$$vz = 0, \qquad z^2 = -1. \tag{10}$$

It is easy to see that such a representation is unique. The set of all vectors z satisfying relations (10) is geometrically equivalent to a Euclidean unit sphere.

The kernel of the operator $\hat{U}$ depends on two vectors w and w'. By analogy with (9), w' may be represented in the form

$$w' = v \operatorname{ch} s' + z' \operatorname{sh} s'. \tag{11}$$

The vector v in formulas (9) and (11) is one and the same, since v is a constant of the motion. Consider the triangle wvw'. If σ is the length of the side ww', then, according to (1), (9), and (11),

$$\operatorname{ch} \sigma = ww' = \operatorname{ch} s \cdot \operatorname{ch} s' - \operatorname{sh} s \cdot \operatorname{sh} s' \cdot \cos \alpha. \tag{12}$$

Here α denotes $\angle wvw'$. Obviously, $\cos \alpha = -zz'$. Using formula (2), we represent relation (12) in the form

$$4 \operatorname{sh}^2 \frac{\sigma}{2} = -(w - w')^2. \tag{13}$$

In the nonrelativistic limit (s, $s' \ll 1$), according to (3), $v_{\text{rel}} = s$. In this case expression (13) turns out to be equal to $(\vec{v} - \vec{v}')^2$, where $\vec{v}$ and $\vec{v}'$ are two values of the three-dimensional relative velocity. Hence it is clear that the quantity (13) is a suitable relativistic analogue for the argument of the kernel of the operator $\hat{U}$. The equation for the ψ-function of relative motion $\psi(w) \equiv \psi(s, z)$ can be written in the form

$$\hat{M}\psi \equiv M_0(s)\psi(s,z) + \int U\left(4\,\mathrm{sh}^2\frac{\sigma}{2}\right)\psi(s',z')\,\mathrm{sh}^2 s'\,ds'\,d\Omega = M'\psi(s,z). \quad (14)$$

Here M' is the eigenvalue of the total mass, and the quantity $\mathrm{sh}^2 s'\,ds'\,d\Omega$ is the volume element of velocity space in the variables s' and z'. Equation (14) satisfies all the requirements posed. Its spectrum may be either discrete ($M' < m_1 + m_2$) or continuous ($M' > m_1 + m_2$). In the nonrelativistic limit, equation (14) goes over into the corresponding nonrelativistic Schrödinger equation describing the relative motion of two particles in the v-representation. From the form of equation (14) it is clear that its internal symmetry corresponds to the group of three-dimensional rotations (in the space of the vectors z). Therefore the solutions of equation (14) can be classified by the eigenvalues of the mass M' and of the total angular momentum L in the center-of-mass system (spin), which is in full agreement with the general theory of irreducible representations of the full Lorentz group developed by Yu. M. Shirokov [3].

In conclusion, we note that equation (14) is only one of the examples illustrating the path outlined here for the relativistic generalization of quantum mechanics. Further development of this direction should lead to a scheme as general as quantum field theory.

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Note: Figure translations are in progress. See original paper for figures.