G. I. GUREVICH

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Abstract

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CONTINUUM MECHANICS

G. I. GUREVICH

ON THE THEORY OF OSCILLATIONS OF SMALL AMPLITUDE

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In the propagation of oscillations of small amplitude in metals, rocks, inorganic glasses, and high polymers, one observes an almost independent dependence of the logarithmic decrement of damping Δ on the oscillation frequency f *. This phenomenon can be interpreted by taking into account that, in its regularity, non-Hookean deformation does not depend on the specific microstructure of different bodies **. Hookean deformations e_n (related by Hooke' s law to the stresses σ_n) are determined by the deviations, at a given moment, of the mean distances along the corresponding principal axes between neighboring structural units of the body (atoms or molecules) from the equilibrium distance corresponding to the absence of load. Non-Hookean deformations of an "infinitely small" (macroscopic) parallelepiped of the body, on the other hand, are a consequence of changes with time in the numbers m_1, m_2, m_3 of the indicated structural units along its edges (with their total number $M = m_1 m_2 m_3$ within the parallelepiped remaining unchanged), which macroscopically is expressed in a coordinated change in the lengths of these edges. The change in the numbers m_1, m_2, m_3 occurs as a result of rearrangements of various elements of the body's microstructure (hereafter we shall call them microparticles, or simply particles), formed by atoms and molecules. In this process the volume of the parallelepiped also changes as a result of a change in the packing density of the microparticles. The features of the mechanism of their rearrangement are manifold and differ in different bodies (we consider only materials that are practically homogeneous and isotropic and are in the solid state). However, the macroscopic result of rearrangements of microparticles will be one and the same: a change in the dimensions of the elementary parallelepiped of the body. Rearrangements of microparticles (by this general term we denote both their displacements and the diffusional "jumps" of structural units, as well as so-called dislocations, etc.) are a consequence of fluctuations of thermal energy, leading to a new arrangement of neighboring microparticles with a smaller deviation of the mean distances between the structural units composing them from the equilibrium value (and with the release of the corresponding elastic energy).

An averaged model of an elementary volume of the body leading to the same macroscopic result may be a system of identical particles connected by elastic bonds, with neighboring particles from time to time rearranging into an equilibrium position. If

* In a summary of data ([1]) confirming the generality of this regularity for a wide range of materials, an exception is made for ferromagnetic metals and high polymers, since for them a typical dependence of Δ on f is considered to be one characterized by the presence of maxima of Δ at the corresponding f. However, in studying these materials one usually ignores the damping observed, for example, in α-iron free of impurities, where Δ depends very little on f ([2]). As for high polymers, for example, in ([3,4]) an almost independent dependence of Δ on f is shown for longitudinal, transverse, and Rayleigh waves in polymethyl methacrylate. At the same time, under large loads, when the specificity of the macromolecular structure of polymethyl methacrylate becomes manifest, one obtains a dependence of Δ on f with the presence of maxima and minima of Δ.

** Attempts to interpret the weak dependence of Δ on f have also been made on the basis of other initial premises (for example ([1,5])).

they completely lose their connection with their former positions, then irreversible (residual) deformations $\varepsilon_{\text{res},n}$ arise. Then from the model it follows ([6])

$$\frac{d\varepsilon_{\text{res},n}}{dt} = \frac{(e_n - \theta_\Gamma/3)}{T_{\text{res}}} \qquad (n = 1, 2, 3), \tag{1}$$

where $\theta_\Gamma = \sum_{n=1}^{3} e_n$. In ([6]) expression (1) was obtained without taking into account the fact that a different number of particles takes part in the rearrangement acts depending on the fluctuation energy U. If this is taken into account, we obtain

$$\frac{d\varepsilon_{\text{res},n}}{dt} = \int_{U_0}^{NU_0} \left[\frac{(e_n - \theta_\Gamma/3)}{T_{\text{res}}}\right] \exp\left\{-\frac{U - U_0}{k\vartheta}\right\} \frac{dU}{k\vartheta}, \tag{2}$$

where U_0 is the energy expended on average per one particle of the model participating in a rearrangement; k is Boltzmann' s constant; ϑ is the temperature; N is the number of all particles of the model. Introducing $T^*_{\text{res}} = T_{\text{res}} \exp\{(U - U_0)/k\vartheta\}$, we transform (2) to the form

$$\frac{d\varepsilon_{\text{res},n}}{dt} = \int_{T_{\text{res}}}^{\infty} \frac{d\varepsilon^*_{\text{res},n}}{dt} \frac{dT^*_{\text{res}}}{T^*_{\text{res}}} = \int_{T_{\text{res}}}^{\infty} \frac{e_n - \theta_\Gamma/3}{T^*_{\text{res}}} \frac{dT^*_{\text{res}}}{T^*_{\text{res}}} \tag{3}$$

(here NU_0 may be replaced by ∞). Thus, $d\varepsilon_{\text{res},n}/dt$ is the sum of the velocities $(e_n - \theta_\Gamma/3)/T^*_{\text{res}}$, distributed over relaxation times T^*_{res} with density $1/T^*_{\text{res}}$, which, as we see, is a direct consequence of the Boltzmann distribution of fluctuations in energy. Since in (3) $e_n - \theta_\Gamma/3$ does not depend on T^*_{res}, integration gives the former result (1)*.

The situation is different with the component e_y of the non-Hookean deformation, which is a consequence of fluctuations of small energy, incapable of producing irreversible rearrangements, but causing very small reversible displacements of microparticles associated with the expenditure of energy $U_{0,y} \ll U_0$ per particle of the corresponding averaged model participating in the act of displacement. We shall call e_y the main component of the deformation of elastic aftereffect. Unlike the additional ("one-component") components, it is caused not by specific factors of inhomogeneity (admixtures of C or N in iron, the role of the macromolecular structure, etc.), but by imperfections of contact of microparticles inherent in all solid bodies. To these imperfections there corresponds a practically continuous spectrum of values U (in the interval $U_{0,y} \ll U < U_0$) required for the above-mentioned small displacements of microparticles (possible as a consequence of the indicated contact imperfections). After the removal of the external load on an elementary volume of the body encompassing the microparticles, they return, under the action of "reverse stresses" $\sigma_{y,n}$ ([12]) and as a consequence of the same fluctuations with $U < U_0$, to their former positions. For $de_{y,n}/dt$, analogously to the preceding, we obtain

$$\frac{de_{y,n}}{dt} = \int_{T_y^*}^{T_M} \left(\frac{de_{y,n}^*}{dt}\right) \frac{dT_y^*}{T_y^*} \qquad (n = 1, 2, 3), \tag{4}$$

Having written that the sum of $(e_n - \theta_\Gamma/3)/T_{\text{res}}$ and de_n/dt is equal to $d\varepsilon_n/dt$

(the full deformation rate), we obtain (expressing e_n through σ_n) the three-dimensional Maxwell equation. Taking account of the dependence of T_{res} on σ_n, it may serve as the basis for solving a wide range of problems in the strength of materials and hydromechanics ([6–10]). It is not applicable to the derivation of equations for small-amplitude oscillations, since, in the case of small σ_n, the values of T_{res} in solid bodies are so large that ε_{res} is practically absent, and the deformation e_y considered below acquires decisive importance (in problems of acoustics and seismics, where very small displacements of points of the body are essential), as do also, in special cases, the additional components of the deformation of elastic aftereffect.

where $T_M = T_y \exp\{(U_0 - U_{0,y})/k\vartheta\}$ (with $U_0/U_{0,y} \gg 1$ and $\ln(T_M/T_y) \gg 1$), and $de_{y,n}^*/dt$ is determined from

$$\frac{de_{y,n}^*}{dt} = \frac{\mu(e_n - \theta_\Gamma/3) - \mu_y(e_{y,n}^* - \theta_y^*/3)}{T_y^*(\mu + \mu_y)} + \frac{K\theta_\Gamma - K_y\theta_y^*}{3T_y^*(K + K_y)}, \tag{5}$$

where $\theta_y^* = \sum_{n=1}^{3} e_{y,n}^*$, and μ and K are the Hookean moduli of shear and volume expansion; μ_y and K_y are the corresponding relaxation moduli, which we assume do not depend on T_y^*.* In the case of a practically absent other deformation component, $de_n/dt = de_\eta/dt + de_{y,n}/dt$. From this equation (transformed to the general coordinate system), expressing ε_n in terms of the displacements u_i and taking into account the equations of motion, as well as (5) and (4), we obtain

$$\frac{1}{\rho}\frac{\partial}{\partial t}\left[(\lambda+\mu)\frac{\partial\theta}{\partial x_i}+\mu\nabla^2 u_i\right]=$$

$$=\frac{\partial^3 u_i}{\partial t^3}+\frac{1}{T_y R}\int_{\exp\{-R\}}^{1}\left[\frac{g}{\mu_y^*}\exp\left\{-\frac{gt\xi}{\mu T_y}\right\}\int_0^t \exp\left\{\frac{g}{\mu}\frac{t\xi}{T_y}\right\}\left(\frac{\partial^3 u_i}{\partial t^3}-\frac{K}{\rho}\frac{\partial^2\theta_\Gamma}{\partial t\,\partial x_i}\right)dt\right.$$

$$\left.+\frac{G}{K_y^*}\exp\left\{-\frac{Gt\xi}{KT_y}\right\}\int_0^t \exp\left\{\frac{Gt\xi}{KT_y}\right\}\frac{K}{\rho}\frac{\partial^2\theta_\Gamma}{\partial t\,\partial x_i}\,dt\right]d\xi \qquad (i=1,2,3) \qquad (6)$$

(we have in mind a medium which, before the onset of oscillations, is in an equilibrium state, when at all its points $\sigma_n = \sigma_{y,n}$). In (6), on the left are ordinary quantities, while on the right $\xi = T_y/T_y^*$, $R = \ln(T_M/T_y)$, $g = \dfrac{\mu}{1+\mu/\mu_y^* R}$, $G = \dfrac{K}{1+K/K_y^* R}$. $\mu_y^* = \mu_y/R$ and $K_y^* = K_y/R$ are the total (determined from macroscopic experiment) relaxation moduli.**

The ratios μ_y^*/μ and K_y^*/K macroscopically characterize the degree of compactness of packing of the microparticles, which determines the maximum possible magnitude of their reversible non-Hookean displacements (the smaller the latter, the larger μ_y^*/μ and K_y^*/K).

* Equation (5) is easily obtained in the same way as equation (3.4) in [(12)], differing from (5) by the absence of the last term on the right, which vanishes if the relaxation change of volume may be neglected. The latter was not taken into account in [(12)] because there such a one-component deformation of elastic aftereffect ε_y was considered which is determined by such a large relaxation change of the shape of elementary volumes of the body that the non-Hookean change of their density may be neglected, whereas in the case e_y it is essential both by itself and in combination with the other.

In [(11)] we erroneously assumed that, from the equation for ε_y, with a relaxation modulus negligibly small in comparison with μ (then it formally coincides with (1)), the equation of seismic oscillations follows. In fact, in their propagation the relaxation e_y plays the decisive role. However, in certain problems of the acoustics of a solid body, ε_y can be of substantial significance, and equation (5) (with $e_{y,n}^*$ replaced by $\varepsilon_{y,n}$) is directly applicable in these cases; for example, from it follow the equations for the propagation of oscillations in steel. Taking into account the time dependence of the relaxation on σ_n, it is also applicable to problems of strength of materials [13].

** If $\mu_y^*/\mu \gg 1$ and $K_y^*/K \gg 1$, then θ_Γ in (6) may, in the first approximation, be replaced by $\theta = \sum_{i=1}^{3} \partial u_i/\partial x_i$. In the general case, however, θ_Γ is determined from the same relations from which (6) follows.

and thereby the quantities e_y and Δ. In materials of practically maximum compactness, for example in a quartz single crystal, for which μ_y^*/μ and K_y^*/K are very large, the damping is also very small, along with the negligible value of e_y (14). Another example: at the surface of the Earth, in "loose" rocks of the clay type, Δ is much greater than in "compact" rocks (crystalline and metamorphosed sedimentary ones), but already at a depth of 2-3 km, where all rocks become compact, the values of Δ for all rocks approach one another, and the ratio of the velocities of longitudinal and transverse waves V_p/V_s becomes close to the Hookean value (1.7-1.9), whereas at the surface in loose rocks it is much larger. At the same time Δ_p/Δ_s also changes: from $\Delta_p/\Delta_s \gtrsim 1$ in compact rocks to $\Delta_p/\Delta_s < 1$ in loose ones.

All that has been considered follows from (6). Let, for example, $R = 25$, $T_y = 10^{-8}$ sec, $\mu_y^*/\mu \gg 1$. Then from (6), with $\theta = \theta_r = 0$, substituting $u_2 = u_{2,0} \exp\{(ik_s - \alpha_s)x_1 - i2\pi ft\}$, where $\alpha_s = \Delta_s f/V_s$ is the damping coefficient of a plane transverse wave, we obtain that $\Delta \to 0$ both as $f \to 0$ and as $f \to \infty$, but in the intermediate region—in the interval of variation of f over 9 orders of magnitude ($10^{-2} \lesssim 2\pi f \lesssim 10^{7}$ cps), $\Delta_s = \pi^2\mu/2R\mu_y^*$, i.e. is practically independent of frequency. Under the same conditions

$$\frac{\Delta_p}{\Delta_s} \simeq 1 - \frac{1+\nu}{3(1-\nu)}\left(1 - \frac{K}{K_y^*}\frac{\mu_y^*}{\mu}\right), \tag{7}$$

where ν is the Hookean Poisson coefficient. There are grounds to suppose that, if $\mu_y^*/\mu \gg 1$, then $1 \lesssim (K/K_y^*)(\mu_y^*/\mu) < 3\text{-}4$, i.e. $\Delta_p/\Delta_s \gtrsim 1$. In this case $V_p/V_s \simeq [(\lambda + 2\mu)/\mu]^{1/2}$. Turning to the case of "loose" materials, let us set, for definiteness, $T_y = 10^{-8}$ sec, $R = 25$, $K_y^*/K = 0.5$, $\mu_y^*/\mu = 2 \cdot 10^{-2}$, $\nu = 0.2$. Then from (6), in the interval $10^{-2} \lesssim 2\pi f \lesssim 10^{5}$ cps, we find (the largest and smallest values are indicated, respectively) $V_p/V_s \simeq 4.7\text{-}3.5$, $\Delta_p/\Delta_s \simeq 0.5\text{-}0.4$, $V_p \simeq 2.5\text{-}3.5$ km/sec, $\Delta_s \simeq 0.2\text{-}0.8$, which is close to observational data for rocks of the clay type at depths of the first hundreds of meters (15). Specific factors can modify both the relation between Δ and f, and the entire regularity of propagation of oscillations*. However, it is clear that understanding the basic regularity will also facilitate clarification of the role of specific factors.

Let us note in conclusion that from the initial ideas from which (6) follows, it is apparently possible to obtain also the basic regularity of propagation of oscillations in liquid media, which is associated with taking ε_{res} into account.

Institute of Physics of the Earth named after O. Yu. Schmidt
Academy of Sciences of the USSR

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* It should be borne in mind that in deriving (6) temperature effects are not taken into account (ϑ is regarded as constant); in particular, the role of thermal conductivity, which may be significant under certain conditions, requires separate consideration. In addition, for wavelengths smaller than some minimum, scattering phenomena at the boundaries of microparticles may become significant.

Note: Figure translations are in progress. See original paper for figures.

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