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Abstract

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On a Property of the Orlicz Norm

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Let L_M^* be some Orlicz space of functions defined on a bounded closed set G of finite-dimensional Euclidean space. As I. V. Gelman showed [1], in the study of certain problems of mathematical physics it is convenient to use Orlicz spaces in which the norm has the following property:

$$\lim_{\|u\|_M \to \infty} \frac{1}{\|u\|_M} \int_G M[\, u(x)\,]\, dx = \infty. \tag{1}$$

For the spaces L^p $(p > 1)$, which are a special case of the Orlicz spaces L_M^* determined by the function $M(u) = |u|^p/p$ $(p > 1)$, property (1) is evidently satisfied. It is known, however, that not all properties of L^p spaces remain valid for arbitrary Orlicz spaces. Therefore the question arises under what conditions the Orlicz norm in the space L_M^* has property (1). In the present note an answer to this question is given. (For all definitions and assertions from the general theory of Orlicz spaces used in this note, see [2].)

Lemma. In order that relation (1) be satisfied, it is necessary that for every $k > 0$ the function $M(u)$ satisfy the condition

$$\lim_{u \to \infty} \frac{M(ku)}{uN^{-1}(u)} = \infty, \tag{2}$$

where $N^{-1}(u)$ is the function inverse to the function $N(v)$, complementary to $M(u)$.

Proof. Suppose that condition (2) is not satisfied. Then there will be a $k_0 > 0$, a sequence of numbers u_n $(n = 1, 2, ...)$ tending to infinity, and a $C > 0$ such that

$$\frac{M(k_0 u_n)}{u_{nN}^{-1}(u_n)} \le C \quad (n = 1, 2, ...). \tag{3}$$

Consider sets $G_n \subset G$ with measure $\operatorname{mes} G_n = 1/u_n$ and put

$$u_n(x) = \begin{cases} k_0 u_n, & \text{if } x \in G_n, \\ 0, & \text{if } x \bar{\in} G_n. \end{cases} \quad (n = 1, 2, ...)$$

Using the formula for the norm of a characteristic function, we obtain

$$\|u_n\|_M = k_0 u_n \ \mathrm{mes}\, G_n \, N^{-1}\left(\frac{1}{\mathrm{mes}\, G_n}\right) = k_0 N^{-1}(u_n).$$

Since $\lim_{n\to\infty} N^{-1}(u_n) = \infty$, it follows that

$$\lim_{n\to\infty} \|u_n\|_M = \infty.$$

On the other hand, by virtue of (3),

$$\frac{\int_G M[u_n(x)]\,dx}{\|u_n\|_M} = \frac{M(k_0 u_n)\,\mathrm{mes}\, G_n}{\|u_n\|_M} = \frac{M(k_0 u_n)}{k_0 u_{nN}^{-1}(u_n)} \leq \frac{C}{k_0} \qquad (n = 1, 2, ...),$$

i.e., for the constructed sequence of functions $u_n(x)$ relation (1) is not satisfied. The lemma is proved.

The assertion of the lemma makes it possible to indicate such Orlicz spaces L_M^* for which relation (1) is not satisfied. Such spaces, for example, will be the spaces L_M^* defined by N-functions $M(u)$ complementary to N-functions $N(v)$ satisfying the Δ_3-condition, since for such functions, for large values of u and some k, the inequality

$$M(ku) \leq u N^{-1}(u)$$

holds.

Remark. In checking whether condition (2) is satisfied, instead of the function $N^{-1}(u)$ one may consider the function $\widetilde{N}^{-1}(u)$, inverse to any function $\widetilde{N}(v)$ equivalent to $N(v)$.

Theorem. *In order that relation (1) hold, it is necessary and sufficient that there exist a function $f(u)$ $(0 \leq u < \infty)$ satisfying the condition*

$$\lim_{u\to\infty} f(u) = \infty \tag{4}$$

and such that, for every v and all sufficiently large values of u, the inequality

$$M(uv) \geq u f(u) M(v). \tag{5}$$

holds.

Proof of necessity. If condition (5) is not satisfied, then this means that for every function $f(u)$ satisfying condition (4), there will be found a v_0 and a sequence of numbers $u_n \to \infty$ such that

$$M(u_{nv}0) < u_{nf}(u_n)M(v_0).$$

In particular, such numbers will also be found for the function $f(u) = N^{-1}(u)$. But this means that condition (2), necessary for relation (1) to hold, is not satisfied.

Proof of sufficiency. Suppose that the conditions of the theorem are satisfied. We shall prove first of all that relation (1) is satisfied for any sequence of functions $u_n(x)$ from the subspace E_M, which is the closure in L_M^* of the set of bounded functions.

By $\|u\|_{(M)}$ we denote the norm defined for $u(x) \in L_M^*$ by the equality

$$\|u\|_{(M)} = \inf k,$$

where the infimum is taken over all such $k > 0$ for which

$$\rho\Big(\frac{u}{k}; M\Big) = \int_G M\left[\frac{u(x)}{k}\right] dx \leq 1.$$

At the same time, for functions $u(x) \in E_M$ the equality

$$\int_G M\left[\frac{u(x)}{\|u\|_M}\right] dx = 1.$$

holds.

This norm is equivalent to the ordinary Orlicz norm:

$$\|u\|_{(M)} \leq \|u\|_M \leq 2\|u\|_{(M)}.$$

Let $u_n(x) \in E_M$ $(n = 1, 2, ...)$ and $\lim_{n\to\infty} \|u_n\|_M = \infty$. Then also $\lim_{n\to\infty} \|u_n\|_{(M)} = \infty$. Since

$$\frac{\int_G M[u_n(x)]\, dx}{\|u_n\|_M} \geq \frac{\int_G M[u_n(x)]\, dx}{2\|u_n\|_{(M)}},$$

it is enough to prove that

$$\lim_{n\to\infty} \frac{\displaystyle\int_G M[u_n(x)]\,dx}{\|u_n\|_{(M)}} = \infty.$$

The last relation follows directly from inequality (5). Indeed, for large values of $\|u_n\|_{(M)}$, by (5),

$$\frac{\displaystyle\int_G M[u_n(x)]\,dx}{\|u_n\|_{(M)}} \geq \frac{\|u_n\|_{(M)}\, f(\|u_n\|_{(M)}) \displaystyle\int_G M\left[\frac{u_n(x)}{\|u_n\|_{(M)}}\right]\,dx}{\|u_n\|_{(M)}} = f(\|u_n\|_{(M)}).$$

Now let $u_n(x)$ $(n = 1, 2, ...)$ be an arbitrary sequence of functions from L^*_M $(\|u_n\|_M \to \infty)$. Without loss of generality, we may assume that all $u_n(x) \in L_M$. Since every function from L_M is at distance not exceeding one from E_M, for each $u_n(x)$ one can indicate a function $\tilde{u}_n(x) \in E_M$ such that $\|u_n - \tilde{u}_n\|_M \leq 2$. Moreover, the functions $\tilde{u}_n(x)$ can be chosen so that, for every $x \in G$, the inequality $|\tilde{u}_n(x)| \leq |u_n(x)|$ is satisfied. Then

$$\frac{\displaystyle\int_G M[u_n(x)]\,dx}{\|u_n\|_M} \geq \frac{\displaystyle\int_G M[\tilde{u}_n(x)]\,dx}{\|\tilde{u}_n\|_M + 2},$$

since $\|u_n\|_M \leq \|\tilde{u}_n\|_M + 2$. From the obtained inequality relation (1) follows, since it has already been proved for functions $\tilde{u}_n(x) \in E_M$. The theorem is proved.

Taking in condition (5), as the function $f(u)$, various concrete functions, we obtain various sufficient conditions for the validity of relation (1). Thus, for example, it is easy to verify that the functions $M_1(u) = |u|^\alpha(|\ln|u|| + \beta)$ $\left(\alpha > 1,\ \beta \geq \dfrac{2\alpha - 1}{\alpha(\alpha - 1)}\right)$, $M_2(u) = e^{|u|} - |u| - 1$, $M_3(u) = e^{u^2} - 1$ satisfy condition (5), where as the function $f(u)$ the function $|u|^r$ is taken for some $r > 0$.

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