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WITH ACCOUNT OF
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Abstract

Full Text

MATHEMATICAL PHYSICS

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**SOLUTION OF THE BOUNDARY-VALUE
PROBLEM OF RADIO-WAVE PROPAGA-
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(Presented by Academician N. N. Bogolyubov on 2 January 1961)

The problem considered here is a special case of the solution ⁽¹⁾ for three spherical layers: the 0th layer, $0 \leq r \leq a$, is the Earth of radius a , $\varepsilon_{rr} = \varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} = \varepsilon_0 + i \cdot 4\pi\sigma_0/\omega$, where ε_0 and the conductivity σ_0 do not depend on r , and $\varepsilon_{\theta\varphi} = 0$; the 1st layer, $a \leq r \leq c$, is the ideal atmosphere, $\varepsilon_{rr} = \varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} = 1$, $\varepsilon_{\theta\varphi} = 0$; the 2nd layer, $c \leq r < \infty$, is the ionosphere, regarded as an electron-ion plasma placed in the vertical magnetic field of the Earth H_0 , characterized by the electron concentration $N_e(r)$ and the collision frequency $\nu_{\text{eff}}(r)$ of electrons with other particles.

Taking into account only the motion of the electrons and neglecting the compressibility of the plasma, we write the components of the dielectric-constant tensor as

$$\begin{aligned} \varepsilon_{rr} &= 1 - \frac{v}{1 + is}; & v &= \frac{\omega_0^2}{\omega^2}; & \omega_0^2 &= \frac{4\pi N_e e^2}{m}; \\ \varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} &= 1 - \frac{v(1 + is)}{(1 + is)^2 - u^2}; & u &= \frac{\omega_H}{\omega}; & \omega_H &= -\frac{eH_0}{mc\omega}; \\ \varepsilon_{\theta\varphi} = -\varepsilon_{\varphi\theta} &= \frac{ivu}{(1 + is)^2 - u^2}; & s &= \frac{\nu_{\text{eff}}}{\omega}, \end{aligned} \quad (1)$$

where ω is the frequency of oscillations of the currents exciting the field; c is the speed of light; e and m are the charge and mass of the electron; N_e and ν_{eff} are continuous and differentiable functions of r in the interval (c, ∞) , and in (c, d) they are arbitrary, while for $(r \geq d)$ (the exosphere and outer space) they vary so slowly that, for sufficiently large $r_0 \geq d$, in $(r \geq r_0)$ the approximation of geometrical optics is valid for the wave equations.

As follows from ⁽¹⁾, the problem consists in finding a stationary solution of Maxwell's equations under the corresponding conditions at $r = 0$ and $r \rightarrow \infty$, and the conditions at the discontinuities of the medium for $r = a$ and $r = c$. Wishing to exclude the continuous spectrum χ from consideration, we replace the boundedness condition at $r = 0$ by the condition of absence of waves traveling away from the center of the Earth. The general solution for a Hertz dipole is described by formulas (14') of ⁽¹⁾. We shall now construct a concrete solution for a three-layer medium.

The normal-wave operator L_r , according to (5) and (9) of ⁽¹⁾, will be

$$r^2 \varepsilon_{rr}^k \frac{d}{dr} \left[\frac{1}{\varepsilon_{\theta\theta}^k} \frac{dY^k}{dr} \right] + r^2 k^2 \varepsilon_{rr}^k Y^k - i r^2 k \varepsilon_{rr}^k \frac{d}{dr} \left[\frac{\varepsilon_{\theta\varphi}^k}{\varepsilon_{\theta\theta}^k} Z^k \right] = \chi Y^k, \quad (2)$$

$$r^2 \frac{d^2 Z^k}{dr^2} + r^2 k^2 \tilde{\varepsilon}^k + i r^2 k \frac{\varepsilon_{\theta\varphi}^k}{\varepsilon_{\theta\theta}^k} \frac{dY^k}{dr} = \chi Z^k, \quad k = 0, 1, 2, \quad (3)$$

with the conditions at the boundaries of the layers

$$Y^k|_{r_k} = Y^{k+1}|_{r_k}, \quad Z^k|_{r_k} = Z^{k+1}|_{r_k}, \quad \left. \frac{dY^k}{dr} \right|_{r_k} = \left. \frac{dY^{k+1}}{dr} \right|_{r_k}, \quad \left. \frac{dZ^k}{dr} \right|_{r_k} = \left. \frac{dZ^{k+1}}{dr} \right|_{r_k},$$

$$k = k_1 = \frac{\omega}{c}, \quad \tilde{\varepsilon} = \varepsilon_{\theta\theta} + \frac{\varepsilon_{\theta\varphi}^2}{\varepsilon_{\theta\theta}}, \quad \left. \frac{dY}{dr} \right|_{r_k} = \left[\frac{1}{\varepsilon_{\theta\theta}} \frac{dY}{dr} \right]_{r_k} - ik \left[\frac{\varepsilon_{\theta\varphi}}{\varepsilon_{\theta\theta}} Z \right]_{r_k}. \quad (4)$$

Solutions of equations (2), (3) in layers 0 and 1 are expressed through linear combinations of the spherical Bessel functions $h_\nu^{(1)}(\rho) = \sqrt{\frac{\pi\rho}{2}} H_\nu^{(1)}(\rho)$ and $h_\nu^{(2)}(\rho) = \sqrt{\frac{\pi\rho}{2}} H_\nu^{(2)}(\rho)$; therefore, introducing the two-argument functions

$$D_\nu(x, y) = \frac{i}{2} \begin{vmatrix} h_\nu^{(1)}(kx) & h_\nu^{(2)}(kx) \\ h_\nu^{(1)}(ky) & h_\nu^{(2)}(ky) \end{vmatrix}; \quad D_\nu(x, y') = \frac{i}{2} \begin{vmatrix} h_\nu^{(1)}(kx) & h_\nu^{(2)}(kx) \\ \dot{h}_\nu^{(1)}(ky) & \dot{h}_\nu^{(2)}(ky) \end{vmatrix}; \quad (5)$$

$$D_\nu(x', y) = \frac{i}{2} \begin{vmatrix} \dot{h}_\nu^{(1)}(kx) & \dot{h}_\nu^{(2)}(kx) \\ h_\nu^{(1)}(ky) & h_\nu^{(2)}(ky) \end{vmatrix}; \quad D_\nu(x', y') = \frac{i}{2} \begin{vmatrix} \dot{h}_\nu^{(1)}(kx) & \dot{h}_\nu^{(2)}(kx) \\ \dot{h}_\nu^{(1)}(ky) & \dot{h}_\nu^{(2)}(ky) \end{vmatrix},$$

we write the solution of (2), (3) in the atmospheric layer, satisfying condition (4) at $r_0 = a$ and the radiation condition at $r_{-1} = 0$, in the form

$$Y^{(1)} = C_1 \{D_\nu(r, a') - Z_{yD_\nu}(r, a)\} = C_1 \overline{D_\nu(r, a')}; \quad (6)$$

$$Z^{(1)} = C_2 \left\{ \frac{1}{Z_z} D_\nu(r, a') - D_\nu(r, a) \right\} = C_2 \overline{D_\nu(r, a)}, \quad (7)$$

where $Z_y = \frac{\dot{h}_\nu^{(2)}(k\sqrt{\varepsilon_0}a)}{\sqrt{\varepsilon_0} h_\nu^{(2)}(k\sqrt{\varepsilon_0}a)}$ and $Z_z = \frac{\sqrt{\varepsilon_0} \dot{h}_\nu^{(2)}(k\sqrt{\varepsilon_0}a)}{h_\nu^{(2)}(k\sqrt{\varepsilon_0}a)}$ are the impedances of the Earth for spherical waves of vertical (Z_y) and horizontal (Z_z) polarizations. As $\sigma_0 \rightarrow \infty$, $Z_y \rightarrow 0$, $Z_z \rightarrow \infty$, and Y , Z pass into the corresponding expressions without the overbar. Dots denote derivatives of the functions h_ν with respect to the argument. Equations (2), (3) in the second layer have four independent solutions. In the upper part of the layer ($r > d$) they can be represented by traveling waves in the directions $r \rightarrow 0$ and $r \rightarrow \infty$ of two types: ordinary and extraordinary. Their analytic continuations into the region ($c \leq r \leq d$) determine the four independent solutions of (2), (3) of interest to us. Only two solutions satisfy the condition at $r \rightarrow \infty$, namely those passing into waves escaping to infinity. Denoting them by the symbols o (ordinary) and e (extraordinary), we write the solution of equations (2), (3) in ($r_1 = c \leq r < \infty$):

$$Y^{(2)}(r, \chi) = C_3 Y_e(r, \chi) + C_4 Y_o(r, \chi); \quad (8)$$

$$Z^{(2)}(r, \chi) = C_3 Z_e(r, \chi) + C_4 Z_o(r, \chi). \quad (9)$$

The specific form of (8), (9) for arbitrary functions $N_e^2(r)$ and $\nu(r)$ in the region where interaction occurs, i.e., conversion of o -waves into e -waves and conversely, as well as their reflection, was obtained by the author by the method of numerical integration on a high-speed electronic computer.

In order for the solutions (6), (7) and (8), (9) to turn into eigenfunctions of the operator L_r , they must satisfy the last condition (4) at the point $r_1 = c$. It gives four homogeneous algebraic equations for C_1, C_2, C_3 , and C_4 . The compatibility condition of this system of equations can be written in the form

$$\left[kD_{\nu_j}(a', c') - I_y^e \overline{D_{\nu_j}(a', c)} \right] \left[k\overline{D_{\nu_j}(a, c')} - I_z^o \overline{D_{\nu_j}(a, c)} \right] - \frac{\chi^e}{\chi^o} \left[k\overline{D_{\nu_j}(a', c)} - I_y^o \overline{D_{\nu_j}(a', c)} \right] \left[k\overline{D_{\nu_j}(a, c')} - I_z^e \overline{D_{\nu_j}(a, c')} \right] = 0 \quad (10)$$

where the following notation has been introduced:

$$I_y^e = (\dot{Y}_e/Y_e)_{r=c}, \quad I_z^e = (\dot{Z}_e/Z_e)_{r=c}, \quad \chi^e = (Z_e/Y_e)_{r=c},$$

$$I_y^o = (\dot{Y}_o/Y_o)_{r=c}, \quad I_z^o = (\dot{Z}_o/Z_o)_{r=c}, \quad \chi^o = (Z_o/Y_o)_{r=c}. \quad (11)$$

We shall call $I_y^e, I_z^e, I_y^o,$ and I_z^o the impedances of ordinary and extraordinary waves of vertical and horizontal polarization, and χ^e and χ^o the polarization coefficients of the corresponding waves at the boundary $r = c$. These 6 quantities, which are functions of χ (or of the angle of incidence θ of the wave on the 2nd layer, equal to $\arcsin \frac{\sqrt{\chi}}{kc}$), completely determine the reflection properties of the 2nd layer. Together with the impedances of the Earth's layer Z_y and Z_z , they determine all parameters of the normal waves ν_j, N_j , and $\chi_j = (C_2/C_1)_j$, called the polarization coefficient. The wave numbers ν_j are the roots of the transcendental equation (10). The coefficients χ_j are determined from the solution (4) for $r = c$:

$$\chi_j = k\chi^e \frac{I_z^e - I_z^o \frac{k\bar{D}_{\nu_j}(c', a') - I_y^o \bar{D}_{\nu_j}(c, a')}{I_y^e - I_y^o}}{I_y^e - I_y^o \frac{k\bar{D}_{\nu_j}(c', a) - I_z^o \bar{D}_{\nu_j}(c, a)}{I_z^e - I_z^o}}. \quad (12)$$

The normalizing factor N_j in (11') from (1) may be represented as

$$N_j = C_1^2 \frac{\bar{D}_{\nu_j}(a', c)}{2\nu_j + 1} \left[\frac{\partial f(\nu)}{\partial \nu} \right]_{\nu=\nu_j}, \quad (13)$$

where $f(\nu)$ is the left-hand side of equation (10), represented in the form $k\bar{D}_{\nu}(a', c') - a\bar{D}(a', c)$. Then the expression for the potentials B and A , given in (14') of (1), for the atmospheric layer,

$$B = -\frac{\pi P}{cb^2} \sum_{j=0}^{\infty} \frac{\bar{D}_{\nu_j}(r, a') \bar{D}_{\nu_j}(b, a') (2\nu_j + 1)}{\sin \nu_j \pi \cdot \bar{D}_{\nu_j}(c, a') [\partial f / \partial \nu]_{\nu_j}} P_{\nu_j}[\cos(\pi - \theta)]; \quad (14)$$

$$A = -\frac{\pi P}{cb^2} \sum_{j=0}^{\infty} \frac{\chi_j D_{\nu_j}(r, a) D_{\nu_j}(b, a') (2\nu_j + 1)}{\sin \nu_j \pi \cdot D_{\nu_j}(c, a') [\partial f / \partial \nu]_{\nu_j}} P_{\nu_j}[\cos(\pi - \theta)], \quad (15)$$

where P is the electric moment of the Hertz dipole.

Let us consider several special cases.

I. $H_0 = 0$. Then $\varepsilon_{\theta\varphi} = 0$, and equation (10) for ν_j splits into two:

$$k\bar{D}_{\nu_j}(a', c') - I_y^e D_{\nu_j}(a', c) = 0; \quad (16)$$

$$k\bar{D}_{\nu_j}(a, c') - I_z^o D_{\nu_j}(a, c) = 0. \quad (17)$$

The first gives rise to a spectrum of normal waves of type TH_j , the second of type TE_j . We shall retain this classification for $H_0 \neq 0$, calling the waves quasi- TH or TE , depending on what they become when $H_0 = 0$.

II. **Watson's waveguide case.** $H_0 = 0$, N_e and ν_{eff} are constant in (c, ∞) . Then $\|\varepsilon_2\| = \varepsilon'_2 + i\varepsilon''_2$, and the solutions (8), (9) will be $h_\nu^{(1)}(k\sqrt{\varepsilon_2}r)$. Equation (16) is simplified to

$$\bar{D}_{\nu_j}(a', c') - \frac{1}{\sqrt{\varepsilon_2}} \left\{ \frac{\dot{h}_\nu^{(1)}(k\sqrt{\varepsilon_2}c)}{h_\nu^{(1)}(k\sqrt{\varepsilon_2}c)} \right\} \bar{D}_{\nu_j}(a', c) = 0. \quad (16')$$

Upon replacing $h_\nu^{(1)}(k\sqrt{\varepsilon_0}a)$ by $J_\nu(k\sqrt{\varepsilon_0}a)$, it coincides with Watson's equation ((2^a), p. 561). The inessential difference in Z_y for $4\pi\sigma_0/\omega \gg 1$ is due to the fact that Watson took into account a continuous spectrum associated with $r = 0$. Equation (16') has three branches ν_n : $\nu_i(0, a)$, $\nu_j(a, c)$, and $\nu_k(c, \infty)$, associated respectively with layers 0, 1, and 2. Only $\nu_j(a, c)$, which pass, as $\varepsilon_2 \rightarrow \infty$ and $\varepsilon_0 \rightarrow \infty$, into ideal normal waves of a waveguide with metallic walls $r = a$ and $r = c$, have small attenuation.

III. **The plane waveguide** is obtained by passing in the solution to $a \rightarrow \infty$. We introduce the linear wave number $\bar{\nu} = \nu/a$ and the linear distance $s = a\theta$, and replace the functions $h\nu$ by Debye approximations in region 1 according to Watson ((7), Fig. 22). Then, for $\varepsilon_0 \rightarrow \infty$, we obtain the equation

$$\sin \int_{ka}^{kc} \sqrt{1 - \left(\frac{\nu}{\rho}\right)^2} d\rho - i\varepsilon_2 \left\{ \frac{\sqrt{\frac{k^2\varepsilon_2 - \nu_j^2}{k^2 - \nu_j^2}}}{\sqrt{1 - \left(\frac{\nu}{\rho}\right)^2}} \right\} \cos \int_{ka}^{kc} \sqrt{1 - \left(\frac{\nu}{\rho}\right)^2} d\rho = 0. \quad (16'')$$

Replacing, in the arguments of the trigonometric functions, the phase integral by its approximate expression $(c - a)\sqrt{k^2 - \nu^2}$, we obtain the equation for $\bar{\nu}_j(a, c)$ of paper (8), derived by the authors for a flat earth (see also (11^b)). As $ka \rightarrow \infty$, the spectra of the branches $\bar{\nu}_i(0, a)$ and $\bar{\nu}_k(c, \infty)$, beginning in neighborhoods of the points $\nu = k\sqrt{\varepsilon_0}$ and $k\sqrt{\varepsilon_2}$, become continuous, and the corresponding sums (14) are transformed into integrals identical with contour integrals along the banks of the cuts from the branch points $k\sqrt{\varepsilon_0}$ and $k\sqrt{\varepsilon_2}$. As is known from papers (9, 10), they describe the so-called lateral waves. They are produced by sums of normal waves localized and propagating in the waveguide layers (0, a) and (c, ∞) , and slightly leaking into the main waveguide channel (a, c) . The lateral waves play no role in the far field because of their strong attenuation.

IV. **Diffraction case.** For $H_0 = 0$ and $\varepsilon_2 = 1$, equation (16') is transformed into the form

$$\dot{h}_\nu^{(1)}(ka) - Z_\nu h_\nu^{(1)}(ka) = 0, \quad (16'')$$

which coincides (with the reservation in case II) with Watson's formula⁽²⁶⁾, p. 98). It gives two branches $\nu_j^D(0, a)$ and $\nu_j^D(a, \infty)$, beginning in neighborhoods of the points $\nu = ka\sqrt{\varepsilon_0}$ and $\nu = ka$. The branch $\nu_j^D(a, \infty)$, important for practice, was first investigated by B. A. Vvedenskii in a series of papers⁽³⁾. He used Debye's approximation in region 2 ((7), Fig. 22) for $h_\nu^{(1)}(ka)$ and in region 1 for $h_\nu(k\sqrt{\varepsilon_0}a)$. In this approximation (16''') will be:

$$\tan \left\{ \frac{\pi}{4} + \int_{ka}^{\nu_j} \sqrt{1 - \left(\frac{\nu_j}{\rho}\right)^2} d\rho \right\} = -\frac{i}{\varepsilon_0} \left\{ \sqrt{\nu_j^2 - \varepsilon_0(ka)^2} / \sqrt{\nu_j^2 - (ka)^2} \right\}.$$

It is easily transformed into the phase-integral equation

$$2 \int_{ka}^{\nu} \sqrt{1 - \left(\frac{\nu_j}{\rho}\right)^2} d\rho + i \ln \frac{\sqrt{\nu_j^2 - (ka)^2} - \sqrt{\nu_j^2 - \varepsilon_0(ka)^2}}{\sqrt{\nu_j^2 - (ka)^2} + \sqrt{\nu_j^2 - \varepsilon_0(ka)^2}} = 2\pi \left(j + \frac{1}{4} \right).$$

Eckersley's idea⁽¹¹⁾ of obtaining the phase-integral equation directly from equations (2), (3), as is done in quantum mechanics for self-adjoint operators, was not carried through, since the normal-wave operator L_r ⁽¹⁾ is non-self-adjoint and the theory for constructing the phase-integral equation for it is still lacking.

V. **Sommerfeld case**⁽¹²⁾. We obtain it from the preceding one for $a \rightarrow \infty$, when the branches $\bar{\nu}_j^D(0, a)$ and $\bar{\nu}_j^D(a, \infty)$ pass into continuous spectra, and the sums (14) into the integrals Q_E and Q of paper⁽¹²⁾ along the banks of the cuts from the branch points $\nu = k\sqrt{\varepsilon_0}$ and $\nu = k$. The exception is an isolated eigenvalue of the branch $\bar{\nu}_j^D(a, \infty)$ with wave number $\bar{\nu}_p = k\sqrt{\varepsilon_0}/\sqrt{1 + \varepsilon_0}$, which determines the Zenneck wave.

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Note: Figure translations are in progress. See original paper for figures.

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