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Abstract

Full Text

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On the Number of Prime k-Twins Lying on an Interval of Given Length

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§ 1.

Let, for an integer $k \geq 1$, $\pi_k(x)$ denote the number of pairs of prime numbers $p, p+2k$ belonging to the interval $(0, x)$. Numerous investigations have been devoted to studying the order of growth of the function $\pi_k(x)$, based on the so-called "sieve" method. Along this path, however, it has been possible to obtain only results of the type of upper estimates. The method of trigonometric sums of I. M. Vinogradov (^{1}), as shown in (^{4}), gives asymptotic laws for almost all k. Moreover, for all such k a remainder term of order

$$\ll x(\ln x)^{-3}.$$

was indicated.

It turns out that the latter result can be considerably strengthened. Namely, by I. M. Vinogradov' s method, for almost all k in the expression for $\pi_k(x)$ one obtains a remainder term of order

$$\ll x(\ln x)^{-C},$$

where $C \geq 3$ is an arbitrary constant. More precisely, the following holds:

Theorem 1. *The number $\pi_k(x)$ of pairs of prime numbers $p, p+2k$ (k-twins), lying in the interval $(0, x)$, is expressed by the formula*

$$\pi_k(x) = 2\prod_{p>2}\frac{p(p-2)}{(p-1)^2}\prod_{\substack{p|k\\p>2}}\frac{p-1}{p-2}\int_2^x\frac{dx}{\ln^2 x} + O\left(\frac{x}{\ln^C x}\right)$$

for every integer $2 \leq 2k \leq x(\ln x)^{-C}$, with the exception of no more than

$$\ll x(\ln x)^{-M-C}$$

of them, where $C \geq 3$, $M > 0$ are arbitrary prescribed constants, and p runs through the prime numbers; moreover, the constants in the symbols $\ll$ and O do not depend on k.

We note that for every fixed integer $t \geq 2$

$$\int_2^x \frac{dx}{\ln^2 x} = \frac{x}{\ln^2 x} + \frac{2!x}{\ln^3 x} + \cdots + \frac{(t-1)!x}{\ln^t x} + O\left(\frac{x}{\ln^{t+1} x}\right).$$

It seems a very interesting fact that the first, second, etc. terms of growth of $\pi_k(x)$ have the same dependence on k for

$$2 \leq 2k \leq x(\ln x)^{-C}.$$

Moreover, the indicated bounds for k are natural. In the general case, in the formula for $\pi_k(x)$, instead of

$$\int_2^x \frac{dx}{\ln^2 x}$$

there should stand

$$\int_2^{x-k} \frac{dx}{\ln x \, \ln(x+k)}.$$

The preceding theorem also admits a generalization to arithmetic progressions with the difference of the progression increasing in a certain manner. Namely:

Theorem 2. *Let $\pi_k(x, D)$ be the number of pairs of prime numbers $p, p+2k$ from the interval $(0, x)$, belonging respectively to the progressions $Dn + l'$, $Dm + l''$, where*

$$0 < D \leq (\ln x)^A,$$

where $A > 0$ is arbitrary fixed,

$$1 \leq l', l'' \leq D$$

and l', l'' are relatively prime to D.

Then, for each $2k$

$$2 \leq 2k \leq x/(\ln x)^C, \qquad 2k \equiv l' - l'' \pmod{D},$$

with the exception of no more than

$$\ll x/D(\ln x)^{M+C}$$

of them, where $C \geq 3,\ M > 0$ are arbitrary prescribed constants, the relation holds

$$\pi_k(x,D) = 2\prod_{p>2}\frac{p(p-2)}{(p-1)^2}\prod_{\substack{p|D\\p>2}}\frac{p-1}{p-2}\prod_{\substack{p>2\\p|k,\ p\nmid D}}\frac{p-1}{p-2}\frac{1}{\varphi(D)}\left\{\int_2^x\frac{dx}{\ln^2 x}+O\left(\frac{x}{\ln^C x}\right)\right\},$$

where p runs through the sequence of prime numbers, φ is Euler' s function, and the constants in the symbols $\ll$ and O do not depend on k.

§ 2. We shall confine ourselves to indicating the main points in the derivation of Theorem 2. Introduce the additional notation: $\mu(t)$ is the Möbius function; $d = (D,q)$ is the greatest common divisor of D and q; $g \pmod d$ with the condition

$$qg/d \equiv 1 \pmod d; \qquad N = \frac{qg}{d}; \qquad R(q) = 1,$$

if $(D,q/d) = 1$, and $R(q) = 0$ in the other cases.

Let also, with $\Delta = (\ln x)^\gamma$, $\gamma = 24(M+A+3C+1)^*$,

$$F_k(x,D) = S(k,x)\sigma_\Delta(k,D),$$

where we put

$$S(k,x) = \sum_{3\leq n<x}\ \sum_{3\leq m<x}\ \sum_{k=n-m}\frac{1}{\ln n\cdot\ln m},$$

$$\sigma_\Delta(k,D) = \sum_{1<q<\Delta}R(q)\frac{\mu^2(q/d)}{\varphi^2(q/d)}\sum_{\substack{a=1\\(a,q)=1}}^{q}\exp\left[2\pi i\,\frac{a}{q}\{N(l'-l'')-k\}\right].$$

In this notation, by the method of trigonometric sums of I. M. Vinogradov, in the form of the work of N. G. Chudakov (^{2}), we obtain the fundamental lemma:

$$\sum_{|k|<x-3}\left|\varphi^2(D)\pi_k(x,D)-F_k(x,D)\right|^2 \ll x^3(\ln x)^{-M-3C},$$

where $\ll$ depends only on A, M, and C. Now from the sequence of integers k that are determined by the conditions $2 \leq 2k \leq x(\ln x)^{-C}$, $2k \equiv l' - l'' \pmod{D}$, we form two subsequences.

To the first of them we assign the numbers for which the inequality holds

$$\left|\varphi^2(D)\pi_k(x, D) - F_k(x, D)\right| > xD(\ln x)^{-C}.$$

Let $V(x, D)$ be the number of terms of the first subsequence. Then from the fundamental lemma we conclude that

$$V(x, D) \ll xD^{-1}(\ln x)^{-M-C},$$

and for each number k of the subsequence complementary to the first, the relation is fulfilled

$$\pi_k(x, D) = \frac{F_k(x, D)}{\varphi^2(D)} + O\left(\frac{xD}{\varphi^2(D)\ln^C x}\right). \tag{1}$$

* In work (^{4}), instead of $\theta = 24(M + A + 7)$, it is printed $\theta = 24(M + A + Z)$.

Next we consider the expression $S(k, x)$ and $\sigma_\Delta(k, D)$, which constitute $F_k(x, D)$. We obtain

$$S(k, x) = \int_2^x \frac{dx}{\ln^2 x} + O\left(\frac{x}{\ln^C x}\right), \tag{2}$$

provided only that $2 \leq 2k \leq x(\ln x)^{-C}$. We note that the restriction imposed on the magnitude k is here essentially necessary.

For $\sigma_\Delta(k, D)$, under $2k \equiv l' - l'' \pmod{D}$, one obtains the expression

$$\sigma_\Delta(k, D) = D \sum_{\substack{h=1 \\ (h, Dk)=1}}^{\infty} \frac{\mu(h)}{\varphi^2(h)} \sum_{\substack{t|k \\ (t,D)=1}} \frac{\mu^2(t)}{\varphi(t)} + O\left(\frac{D^2 \ln^3 \ln x \cdot \tau(k)}{\Delta}\right), \tag{3}$$

where $\tau(k)$ is the number of divisors of k.

Although the last estimate is sufficiently sharp, the expression under the symbol O, for $\Delta = (\ln x)^\gamma$, $\gamma = 24(M + A + 3C + 1)$, will not, generally speaking, enter into the remainder term. To choose Δ so that this expression always has the estimate we need, $\ll D(\ln x)^{-C}$, is at present impossible because the laws of distribution of prime numbers in progressions with large difference are unknown.

In this connection additional restrictions must be imposed on k. We shall require that, in addition to the conditions already indicated for k, one more condition be satisfied:

$$\tau(k) < (\ln x)^{M+C}.$$

In this case, from (1)–(3) it follows that

$$F_k(x, D) = \lambda \prod_{p>2} \frac{p(p-2)}{(p-1)^2} \prod_{\substack{p\mid D\\ p>2}} \frac{(p-1)^2}{p(p-2)} \prod_{\substack{p>2\\ p\mid k,\ p\nmid D}} \frac{p-1}{p-2} D \left[\int_2^x \frac{dx}{\ln^2 x} + O\left(\frac{x}{\ln^C x} \right) \right], \tag{4}$$

where $\lambda = 2$ if $2 \nmid D$, and $\lambda = 1$ when $2 \mid D$.

Next we count $U(x, D)$, the number of numbers k from the second subsequence for which

$$\tau(k) > (\ln x)^{M+C}.$$

Here we use a result of A. I. Vinogradov and Yu. V. Linnik [3] on estimates of divisors in progressions, and thus obtain the estimate

$$U(x, D) \ll xD^{-1}(\ln x)^{-M-C}.$$

Consequently, for every integer $2k$

$$2 \leq 2k \leq x(\ln x)^{-C}, \qquad 2k \equiv l' - l'' \pmod{D}$$

(the number of them is equal to $[xD^{-1}(\ln x)^{-C}]$), excluding no more than

$$V(x, D) + U(x, D) \ll xD^{-1}(\ln x)^{-M-C}$$

of them, where $M > 0$ is an arbitrary constant, the asymptotic formula (4) holds. Hence, by virtue of relation (1), theorem 2 follows.

In conclusion we note that, with the aid of I. M. Vinogradov's method, analogous results are also obtained in the more general problem on the number of prime numbers p such that $p + a_1, \dots, p + a_m$, for even $a_1, \dots, a_m$, are also prime.

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Note: Figure translations are in progress. See original paper for figures.

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