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Abstract

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ON THE INCREASE IN THE LENGTH OF A WIND WAVE IN A SHALLOW SEA

(Presented by Academician V. V. Shuleikin, 27 IX 1960)

In solving the problem of forecasting wind waves on shallow seas and lakes, the author advanced a hypothesis ([1,2]) on the applicability, under shallow-sea conditions, of the law of increase of the length of a wind wave derived by V. V. Shuleikin ([3]) for deep-sea conditions, in the form

$$r/R = 1/8 + 0.325\,(R_0/R)^{2/3}, \tag{1}$$

where r is the half-height of the wave; $R_0 = \lambda_0/2\pi$; λ_0 is the initial wavelength; $R = \lambda/2\pi$; λ is the current wavelength. This hypothesis was advanced on the basis of the entirely satisfactory agreement of field data with the theoretical formula (1) ([2]). In the present work we have attempted to substantiate it theoretically as well.

For this purpose we used, as did V. V. Shuleikin, the theorem on the moment of momentum, which may be written in the form

$$d\overline{Q}/dt = \overline{M}, \tag{2}$$

where $\overline{Q}$ is the moment of momentum averaged over one period; $\overline{M}$ is the averaged value of the moment of the external forces; dt is an infinitely small interval of time.

It can be shown, as in work ([4]), that the moment of the external forces in a shallow sea is equal to

$$\overline{M} = (W_v - W_H)/\omega, \tag{3}$$

where W_v is the power supplied to the wave by the wind; W_H is the power lost by the wave owing to the influence of shallow water (W_v and W_H are taken in the calculation per unit surface area of the sea); ω is the angular velocity of a water particle, which, in the case of shallow water, moves along an elliptical orbit. Both components—the positive moment W_v/ω, created by the wind force, and the negative moment $-W_H/\omega$, created by the force of friction against the bottom—are equivalent and completely determine the value of $\overline{M}$. In turn, $W_v - W_H = dE/dt$, where E is the total energy of the wind waves per unit surface area (this equation is the equation of the wave-energy balance ([1])). Hence, instead of (3), one may write

$$\overline{M} = \frac{dE}{dt}\frac{1}{\omega}. \tag{4}$$

On the basis of equations (2) and (4), the theorem on the moment of momentum for a shallow sea is written in the form

$$\omega\, d\overline{Q}/dt = dE/dt. \tag{5}$$

The total energy of the waves per unit surface area of the sea, on the basis of the works of V. V. Shuleikin ([3]) and A. I. Nekrasov ([5]), in the first approxi-

is equal to

$$E \approx \frac{1}{4}\rho g r^2 \left(2 + \frac{r}{R} + 3\frac{r}{R}\,\mathrm{cth}\,\frac{H}{R}\right), \tag{6}$$

where ρ is the density of water; g is the acceleration in the gravitational field; H is the depth of the sea. Differentiating (6) with respect to time, we obtain

$$\frac{dE}{dt} = \rho g r\frac{dr}{dt} + \frac{3}{4}\rho g\frac{r^2}{R}\left(1 + 3\,\mathrm{cth}\,\frac{H}{R}\right)'\frac{dr}{dt} - \tag{7}$$

$$-\frac{1}{4}\rho g\frac{r^3}{R^2}\left(1 + 3\,\mathrm{cth}\,\frac{H}{R}\right)\frac{dR}{dt} + \frac{3}{4}\rho g\frac{r^3 H}{R^3}\frac{1}{\mathrm{sh}^2(H/R)}\frac{dR}{dt}.$$

We shall now find the value $\overline{Q}$. As is known, the trajectory of water particles in their wave motion in shallow water is an ellipse. In Fig. 1 the elliptical orbit of an arbitrary particle S at some depth y is shown schematically. The origin of coordinates here is placed at the center of the orbit of the surface particle, and the y axis is directed downward. Let us consider a particle S with orbit center O at an arbitrary depth y and with phase angle $\angle SOL = \theta$. The angular momentum of an infinitesimal mass $\rho\, dy$, having linear velocity v, relative to the center of moments O_0, is equal to (see Fig. 1)

$$dQ = \rho v\, \overline{O_0 S_0}\, dy. \tag{8}$$

Let us denote the variable distance of the mass S from the center of the orbit O by l, and the angle $\angle S_0 O_0 O$ by θ_1. From consideration of Fig. 1 it is easy to establish that the moment arm $\overline{O_0 S_0}$ is equal to $\overline{O_0 S_0} = y\cos\theta_1 + l\cos(\theta - \theta_1)$.

Fig. 1

Figure 1: Fig. 1

The ellipticity of the orbits of water particles is usually small. Thus, for $H/\lambda = 0.3$ the difference between the minor axis of the ellipse and the major axis is only 4.5% ([3]). Therefore, without great error one may assume that $\cos(\theta - \theta_1) = 1$. Even in the case when $H/\lambda = 0.1$ (at this value of H/λ, waves in shallow water are strongly destroyed), taking $\cos(\theta - \theta_1) = 1$, we introduce an error of at most 15% of the value of l. This error will be still smaller (approximately by half) if we average the quantity $\cos(\theta - \theta_1)$ over a period, since its variation is pulsating in character: four times per period $\cos(\theta - \theta_1)$ takes the value unity (on the axes of the ellipse), and in the intervals between the axes it takes a minimal value, only slightly different from unity. In what follows we shall carry out averaging over θ and over θ_1. Therefore, without making a large error, it is possible to write at once that $\overline{O_0 S_0} \approx y \cos\theta_1 + l$. Note that here it is impossible to replace $\cos\theta_1$ by $\cos\theta$, despite the small difference between θ_1 and θ, because of the significant error arising at small values of the cosines.

Fig. 1

Next, the linear velocity v can be represented in the form $v = \omega l$. Hence, instead of (8), we shall have

$$dQ = \rho\omega l \,(y \cos\theta_1 + l)\, dy. \tag{9}$$

On the basis of the concepts of classical hydromechanics ([6]), the quantity l can be represented in the form

$$l = \frac{r}{\operatorname{sh}(H/R)} \sqrt{\operatorname{sh}^2 \frac{y}{R} + \sin^2\theta}. \tag{10}$$

Substituting (10) into (9):

$$dQ = \rho\omega \frac{r}{\operatorname{sh}(H/R)} \sqrt{\operatorname{sh}^2 \frac{y}{R} + \sin^2\theta} \left(y \cos\theta_1 + \frac{r}{\operatorname{sh}(H/R)} \sqrt{\operatorname{sh}^2 \frac{y}{R} + \sin^2\theta} \right) dy. \tag{11}$$

Integrating (11) from zero to H and averaging over the period, i.e., over θ and θ_1 from zero to 2π, we obtain the desired expression for the averaged momentum of the water masses in shallow water,

$$\overline{Q} = \frac{1}{2}\rho\omega R r^2 \operatorname{cth}(H/R),$$

or, since $\omega R = c$ [3,4]:

$$\overline{Q} = \frac{1}{2}\rho c r^2 \operatorname{cth}(H/R). \tag{12}$$

It follows from (12), for $H \to \infty$ (deep sea), that $\overline{Q}_\infty = \frac{1}{2}\rho c r^2$, and this is the already known expression for the averaged momentum of water masses in deep water (formula (179) from [3]).

Let us differentiate (12) with respect to time and multiply by $\omega = c/R$, taking into account here that $c^2 = gR \operatorname{th}(H/R)$ [6]:

$$\omega \frac{d\overline{Q}}{dt} = \rho g r \frac{dr}{dt} + \frac{1}{4}\rho g r^2 \frac{dR}{dt} + \rho g \frac{r^2}{R^2} \frac{H}{\operatorname{sh}^2(H/R)} \frac{dR}{dt}. \tag{13}$$

Comparing, on the basis of (5), the right-hand sides of (7) and (13), we obtain, after simple transformations and division by $\frac{1}{4}\rho g r^2/R$:

$$\left[1 + \frac{4H/R}{\operatorname{sh} 2(H/R)} + \frac{r}{R}\left(1 + 3\operatorname{cth}\frac{H}{R}\right) - 3\frac{r}{H}\frac{(H/R)^2}{\operatorname{sh}^2(H/R)}\right]\frac{dR}{dt} = 3\left(1 + 3\operatorname{cth}\frac{H}{R}\right)\frac{dr}{dt}. \tag{14}$$

From (14), for $H \to \infty$ (deep sea), we obtain the differential equation whose integral, under certain prescribed initial conditions [3], is written in the form (1)

$$dr/dt = \frac{1}{3}\left(\frac{1}{4} + r/R\right) dR/dt \tag{15}$$

(cf. (15) with formula (191) from [3]). Determining the value of the quantity H/R up to which, say, within an error of 5%, one may use equation (15) instead of the more complicated equation (14), we thereby determine the limits of applicability of formula (1) under shallow-water sea conditions. It turns out that as long as $H/R \gtrsim 2.3$, or $H/\lambda \gtrsim 0.37$, equation (15) may be used, within an error of 5%, for calculating λ from the data $h = 2r$, instead of (14), or, what is the same thing, law (1) may be used. For example, at $H = 3$ m, law (1) will hold within an error of 5% as long as the wavelength has not become approximately 8 m; at $H = 4$ m, correspondingly, $\lambda \approx 11$ m. And precisely these values of λ were attained at wind speeds $v \approx 10$ m/sec at distances on the order of 30 km from the windward shore, with $H \approx 3.5$ m, during storms on Lake Beloye [2]. In other words, our hypothesis, at any rate within these wave-size limits, is confirmed both experimentally and theoretically.

It turns out that also in the case $H/\lambda < 0.37$, although with an error on the order of 10%, formula (1) may be used for calculating λ in a sea of finite depth

Fig. 2

Figure 2: Fig. 2

up to the attainment of the largest possible limiting sizes of wind waves at the given depth. The proof of the latter conclusion is based on a comparison of the curve $\lambda(h)$ for the deep sea (curve H_∞ in Fig. 2), constructed by formula (1) under the initial condition $\lambda_0 \approx 1$ m, $h_0 \approx 0.143$ m ([4]), with curves constructed on the basis of the differential equation (14) by the "field of directions" method ([7]), for depths 3, 4, 6, and 10 m under the initial condition: $H/\lambda \approx 0.37$ (Fig. 2). For example, at $h = 1.5$ m, by formula (1) we have $\lambda_\infty \approx 29.5$ m, while from the curve $H = 3$ m we have $\lambda_H \approx 26.5$ m. In this case the error in determining λ_H will be maximal, on the order of 10%. At smaller h

the error will, naturally, be smaller; moreover, if one uses the curve H_∞ (or formula (1)) to compute the wavelength in a shallow sea, the error will be in the direction of overestimating the value of λ_H. Formula (1) may be used for $H/\lambda < 0.37$ within an error of 5%, if a correction equal to $-0.05\lambda_\infty$ is introduced, where λ_∞ is the wavelength computed from (1) or found from the curve H_∞ in Fig. 2.

Let us illustrate this with a numerical example. Suppose that at a depth $H = 3$ m a wave height $h = 1.5$ m has been measured. From formula (1), or from the curve H_∞, we find the corresponding wavelength, equal to $\lambda_\infty \approx 29.5$ m. To an accuracy of 5% of the computed quantity, one may assume that the true wavelength is

$$\lambda_H \approx 29.5 \text{ m} - 0.05\,\lambda_\infty \text{ m} \approx 28 \text{ m}$$

(in fact, on the basis of the curve $H = 3$ m it is equal to $\lambda_H \approx 26.5$ m).

It can be shown that for $H/\lambda \gtrsim 0.5$ equations (14) and (15) almost completely coincide. Thus, for calculating the wavelength from its given height h, one may, within an error of $0.05\lambda_\infty$, use formula (1) when $H/\lambda \gtrsim 0.37$, and when $H/\lambda < 0.37$, with the same accuracy, if a correction of the form $-0.05\lambda_\infty$ is introduced. The minus sign in this correction indicates that the wavelength in a shallow sea cannot be greater than its length in a deep sea for one and the same wave height: shallow-water waves are somewhat steeper than deep-water waves.

Fig. 2

The greatest possible wave height at a sea depth H determines the corresponding limiting wavelength. Therefore the curves $\lambda(h, H)$ in Fig. 2 break off at definite points corresponding to the abscissa $h = H/2$ (according to observational data, at depth $H \approx 2h$ an intensive destruction of wind waves occurs under the influence of shallow water).

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CITED LITERATURE

1. A. P. Khan, *Izv. AN SSSR, ser. geofiz.*, No. 10 (1958).

2. A. P. Khan, *Izv. AN SSSR, ser. geofiz.*, No. 1 (1961).

3. V. V. Shuleikin, *A Short Course in the Physics of the Sea*, L., 1959.

4. V. V. Shuleikin, *Tr. Morsk. gidrofiz. inst. AN SSSR*, **9** (1956).

5. A. I. Nekrasov, *Exact Theory of Waves of Steady Form on the Surface of a Heavy Fluid*, *Izv. AN SSSR*, 1951.

6. G. Lamb, *Hydrodynamics*, 1947.

7. M. Ya. Vygodsky, *Handbook of Higher Mathematics*, Moscow, 1956.

Note: Figure translations are in progress. See original paper for figures.