P. E. Dyubyuk

MATHEMATICS

1961

Abstract

Full Text

P. E. Dyubyuk

ON THE NUMBER OF SUBGROUPS OF A FINITE ABELIAN GROUP

(Presented by Academician A. I. Mal' tsev on 24 X 1960)

MATHEMATICS

§ 1. In the present paper we consider the question of the number of subgroups of a given order of the various types of finite abelian p-groups.

The paper uses the following definitions, previously introduced by the author ([1]).

Let p be a prime number, and let α and β be integers. The symbol $P_{\alpha,\beta}$ is defined as follows: $P_{\alpha,\beta} = (p^\alpha - 1)(p^{\alpha-1} - 1)\cdots(p^{\beta+1} - 1)$, if $0 \leqslant \beta < \alpha$; $P_{\alpha,\beta} = 1$, if $\beta = \alpha$ or $\alpha \leqslant \beta = 0$; $P_{\alpha,\beta} = 0$ in all other cases, i.e., if $0 > \beta \neq \alpha$ or $\alpha < \beta \neq 0$.

Put

$$\varphi_{\alpha,\rho} = \frac{P_{\alpha,\alpha-\rho}}{P_{\rho,0}} = \frac{P_{\alpha,\rho}}{P_{\alpha-\rho,0}}.$$

It follows from this definition that, if the condition $\alpha \geqslant \rho \geqslant 0$ is not satisfied, then $\varphi_{\alpha,\rho} = 0$ for $\alpha \neq \rho$ and $\rho \neq 0$; $\varphi_{\alpha,\rho} = 1$ for $\rho = 0$ or $\alpha = \rho$.

Let $d_1, d_2, \dots, d_l$ be nonnegative integers, and suppose that for every $i = 1, 2, \dots, l-1$ we have $d_i \geqslant d_{i+1}$; let $a_1, a_2, \dots, a_l$ be arbitrary integers. Define the symbol

$$\left\{ \begin{matrix} d_1, d_2, \dots, d_l \\ a_1, a_2, \dots, a_l \end{matrix} \right\} = \left\{ \begin{matrix} d_i \\ a_i \end{matrix} \right\}_{i=1}^{l}$$

as follows:

$$\left\{ \begin{matrix} d_i \\ a_i \end{matrix} \right\}_{i=1}^{l} = p^{\sum_{i=1}^{l}(d_i - a_i)a_{i+1}} \prod_{i=1}^{l} \varphi_{d_i - a_{i+1}, a_i - a_{i+1}}, \qquad a_{l+1} = 0.$$

Let P be an arbitrary abelian p-group, d its rank, and p^k the greatest of the orders of the elements of the group P. We shall agree to denote by d_i the number of elements of a minimal basis of the group P whose orders are not less than p^i. Thus, obviously, $d_1 = d$, $d_{k+1} = 0$, $d_{k+2} = 0$, etc. The numbers d_i $(i = 1, 2, \dots)$

will be called the **invariants** of the group P, and we shall say, with respect to the group P itself, that it is of type $\{d_1, d_2, \dots, d_k\}$.

In deriving the theorems formulated below, the following propositions, proved by the author in ([1]), are used.

Lemma A. If a, b, c are nonnegative integers, then the identity

$$\sum_{i=0}^{a} p^{i(b-c+i)} \varphi_{b,c-i} \varphi_{a,i} = \varphi_{a+b,c}.$$

Theorem A. Let P be an abelian p-group of type $\{d_1, d_2, \dots, d_k\}$. Let $a_1, a_2, \dots, a_k$ be integers. The number of subgroups of type $\{a_1, a_2, \dots, a_k\}$ of the group P is equal to

$$\begin{Bmatrix} d_i \\ a_i \end{Bmatrix}_{i=1}.$$

§ 2.

A direct application of Theorem A makes it possible to prove the following proposition:

Theorem 1. Let P be an abelian group of order p^n, of rank 2, and of type $\{d_1, d_2, \dots, d_m\}$. If $d_l = 2,\ d_{l+1} < 2$ $(l \le m)$, then the number of subgroups of order p^α

$$\left(\alpha \le \left[\frac{n+1}{2}\right]\right)$$

of the group P is equal to $\varphi_{\sigma+1,1}$, where σ is the smaller of the numbers α and l.

Further, using Lemma A, Theorem A, and applying induction on s, one can justify the following result:

Theorem 2. Let P be an abelian group of type $\{d_1, d_2, \dots, d_s, d_{s+1}, \dots, d_r\}$, with $d_1 = d_2 = \dots = d_s > d_{s+1}$. The number of subgroups of order p^α $(0 \le \alpha \le s)$ of the group P is equal to $\varphi_{d_1+\alpha-1,\alpha}$.

Applying the same apparatus, as well as Theorem 2, we prove the following theorem:

Theorem 3. Let P be an abelian group of order p^n, of type $\{d_1, d_2, \dots, \dots, d_{s-1}, d_s\}$, with $d_1 = d_2 = \dots = d_{s-1}$ (so that $n = (s-1)d_1 + d_s$). The number of subgroups of order p^α $(0 \le \alpha \le n)$ of the group P is congruent to $\varphi_{\alpha+d_1-1,\alpha}$ modulo $p^{n-\alpha+1}$.

In Theorem 10 of the work (1) an estimate is given for the number of subgroups of order p^α of an arbitrary abelian group. This estimate is given by means of the function $\varphi_{n,\alpha}$. It is interesting to note that the estimate of the number of subgroups of order p^α of the group P (characterized in the hypothesis of Theorem 3) cannot be improved by introducing any third function of the form $\varphi_{l,\alpha}$. Indeed, the following proposition holds:

Theorem 4. Let P be the group defined in the hypothesis of Theorem 3, and let N_α be the number of subgroups of order p^α of the group P $(0 \le \alpha \le n)$. If $d_1 \ne n-\alpha+1$ and σ is greater than each of the numbers d_1 and $n-\alpha+1$, then for no l can the congruence $N_\alpha \equiv \varphi_{l,\alpha} \pmod{p^\sigma}$ hold.

Relying on Theorem 3, one can prove the following theorems:

Theorem 5. Let P be an abelian group of order p^n, of type $\{d_1, d_2, d_3\}$, so that $n = d_1 + d_2 + d_3$. The number of subgroups of order p^α $(0 \le \alpha \le n)$ of the group P is congruent to $\varphi_{d_1+\alpha-1,\alpha}$ modulo p^σ, where σ is the smaller of the numbers $d_1 + d_2 - 1$ and $n - \alpha + 1$.

Theorem 6. Let P be an abelian group of order p^n, of type $\{d_1, d_2, d_3, d_4\}$, so that $n = 2d_1 + d_3 + d_4$. The number of subgroups of order p^α $(0 \le \alpha \le n)$ of the group P is congruent to $\varphi_{d_1+\alpha-1,\alpha}$ modulo p^σ, where σ is the smaller of the numbers $2d_1 + d_3 - 2$ and $n - \alpha + 1$.

Theorem 7. Let P be an abelian group of order p^n, of type $\{d_1, d_2, d_3, d_4\}$, so that $n = d_1+d_2+d_3+d_4$. The number of subgroups of order p^α $(0 \le \alpha \le d_1+2)$ of the group P is congruent to $\varphi_{d_1+\alpha-1,\alpha}$ modulo p^σ, where σ is the smaller of the numbers $d_1 + d_2 - 1$ and $n - \alpha + 1$.

It should be noted that in proving the last theorems one has to overcome great and ever-increasing technical difficulties, so that the derivation of each of these propositions takes many pages.

On the other hand, it is clear that one can formulate a proposition of a more general character, pertaining to any abelian group, although in deriving it one would have to overcome very great difficulties. This is the following **conjectured result**:

Let P be an abelian group of order p^n and of type $\{d_1, d_2, \dots, d_k, d_{k+1}, \dots, d_s\}$, with

$$d_2 = d_3 = \cdots = d_k = d_1$$

and

$$d_{k+1} < d_1 \qquad \left(\text{so that } n = kd_1 + \sum_{i=k+1}^{s} d_i\right).$$

$(1 \leq k \leq s)$. The number of subgroups of order p^α $(0 \leq \alpha \leq n)$ of the group P is congruent to $\varphi_{d_1+\alpha-1,\alpha}$ modulo p^σ, where σ is the smaller of the numbers $\sum_{i=1}^{k+1} d_i - k$ and $n - \alpha + 1$.

It is not difficult to see that Theorems 5, 6, and 7 are special cases of the proposition just stated. It is easy to show, using Theorem 2, that Theorem 3 is also a special case of this latter, far-reaching result.

Received
22 IX 1960

REFERENCES

1. P. E. Dyubyuk, *Izv. AN SSSR, ser. matem.*, **12**, No. 4, 351 (1948).

Note: Figure translations are in progress. See original paper for figures.