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Abstract

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MATHEMATICS

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ON THE COMPARISON OF THE COMPLEXITY OF IRREDUNDANT D.N.F.' s AND MINIMAL D.N.F.' s

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1. This note concerns the theory of disjunctive normal forms[(1)] (d.n.f.' s). It studies the relation between the complexity of an arbitrary irredundant d.n.f. and that of a minimal d.n.f. realizing one and the same function of the algebra of logic. This question is of interest in connection with problems of finding minimal d.n.f.' s. Since algorithms for finding minimal d.n.f.' s require a large enumeration[(2)], in practice one often simplifies a d.n.f. by means of some set of transformations, and the result of this simplification, generally speaking, depends on the sequence of transformations. Usually, for each sequence of transformations the process breaks off at a finite step and leads to an unsimplifiable d.n.f., called an irredundant d.n.f. Minimal d.n.f.' s are among the irredundant ones. Since the number of irredundant d.n.f.' s grows very rapidly with the growth of the number of variables of the functions realized by them, choosing a minimal d.n.f. from among the irredundant ones requires very large enumeration.

The question arises to what extent, in finding a minimal d.n.f., it is justified to restrict oneself to obtaining some irredundant d.n.f. Among nonmathematicians using the apparatus of d.n.f.' s, the opinion is widespread that an arbitrary irredundant d.n.f. is minimal or "almost minimal." In this paper the erroneousness of such opinions is shown.

The main results concern two directions. In the set of all functions of n variables, such functions are constructed for which some irredundant d.n.f. is $2^{n-c\sqrt{n}}$ times "more complex" than a minimal d.n.f. At the same time it is shown that in a certain set of "almost all" functions of n variables, for any function from this set an arbitrary irredundant d.n.f. cannot be "more complex" than a minimal one by more than $2n$ times, and functions are constructed on which this estimate is attained.

2. We introduce the following notation and definitions. x^n is a tuple $(x_1, \dots, x_n)$; x^σ denotes x for $\sigma = 1$ and $\overline{x}$ for $\sigma = 0$; $\Phi(f(x^n))$ is a formula Φ realizing a function f of n variables. By conjunctions we shall mean expressions of the form $\bigwedge_i x_i^{\sigma_i}$ and the functions of the algebra of logic realized by them. The expressions $x_i^{\sigma_i}$ will be called factors of the conjunction $\bigwedge_i x_i^{\sigma_i}$. By the symbol $X \in \Phi$ we denote that the conjunction X occurs in the d.n.f. Φ.

$r(K)$ is the rank of the conjunction K (the number of factors in K); $\mathrm{Dim}(K) = n - r(K)$, where n is the number of variables of the functions of which the conjunction K is considered (i.e., $\mathrm{Dim}(K)$ is the number of inessential variables in the conjunction K). We shall say that the conjunctions K and Q intersect if $K \cdot Q \not\equiv 0$.

We shall say that a function φ absorbs a function ψ if $\varphi \cdot \psi = \psi$. In this case the set of assignments on which $\psi = 1$ is a subset of the set of assignments on which $\varphi = 1$. A conjunction X absorbed by a function f will be called minimal relative to f if the conjunction X_i—the result of deleting the variable x_i from X —is not absorbed by the function f. A conjunction will be called minimal if it is minimal with respect to all variables entering it.

A DNF $T(f)$ will be called irredundant if: a) all conjunctions K in $T(f)$ are minimal relative to f; b) each conjunction occurs in $T(f)$ only once and no conjunction K from $T(f)$ is absorbed by the disjunction of the remaining conjunctions from $T(f)$.

$I(T)$ is the number of conjunctions in the DNF T;

$$Y(f) = \max_{T_1, T_2} [I(T_1)/I(T_2)],$$

where the maximum is taken over all pairs T_1, T_2 of irredundant DNFs of the function f;

$$Y(\mathfrak{M}) = \max_{f \in \mathfrak{M}} Y(f),$$

where $\mathfrak{M}$ is the given set of functions f;

$$Y(n) = \max_{f(x^n)} Y(f(x^n)).$$

3. Let functions $f_1(x^k)$ and $f_2(y^l)$ be given. Define the function

$$\pi = \pi(x^k, y^l, z, w) = f_1(x^k) z \overline{w} \vee \overline{f_1}(x^k) \overline{z} w \vee f_2(y^l) \overline{z}\overline{w} \vee \overline{f_2}(y^l) z w. \qquad (1)$$

Define

$$\pi' = \pi'(x^k, y^l, z, w) = \pi(x^k, y^l, \overline{z}, \overline{w}); \qquad \pi' = \overline{\pi}.$$

It turns out that, knowing one irredundant DNF for each of the functions $f_1(x^k), \overline{f_1}(x^k), f_2(y^l), \overline{f_2}(y^l)$, respectively T_x, T'_x, T_y, T'_y, one can construct two irredundant DNFs for π.

In accordance with (1), define a DNF $A(\pi)$ of the function π:

$$T_{xz}\overline{w} \vee T'_x\overline{z}w \vee T_y\overline{zw} \vee T'_{yzw}. \tag{2}$$

The DNF $A(\pi)$ consists of conjunctions of the form: $Xz\overline{w}$, $X'\overline{z}w$, $Y\overline{zw}$, $Y'zw$, where X, X', Y, Y' are conjunctions from the DNFs T_x, T'_x, T_y, T'_y, respectively.

Define a DNF $B(\pi)$ of the function π:

$$\bigvee_{\substack{X\in T_x\\ Y\in T_y}} XY\overline{w} \vee \bigvee_{\substack{X\in T_x\\ Y'\in T'_y}} XY'z \vee \bigvee_{\substack{X'\in T'_x\\ Y\in T_y}} X'Y\overline{z} \vee \bigvee_{\substack{X'\in T'_x\\ Y'\in T'_y}} X'Y'w. \tag{3}$$

Lemma 1. The DNFs $A(\pi)$ and $B(\pi)$ are irredundant.

4. Returning to (1), (2), and (3), call the functions f_1 and f_2 the component functions of π, and the DNFs T_x, T'_x, T_y, T'_y the component DNFs of $A(\pi)$ and $B(\pi)$. We shall call the linear function

$$L(u^s) = u_1 + \dots + u_s \pmod 2$$

a function of order 0 and denote it by π^0. We shall call a function π a function of order p and denote it by π^p if its component functions $f_1(x^k)$ and $f_2(x^l)$ are functions of order $p-1$, with $|k-l| \le 1$. For any $p \ge 0$ and $n \ge n_p = 3 \cdot 2^p - 2$, there exist functions of order p in n variables. The linear functions $L(u^s)$ and $\overline{L}(u^s)$ have only one DNF (which is at the same time an irredundant DNF). Therefore the component DNFs $A(\pi^1)$ and $B(\pi^1)$ for the function π^1 are determined uniquely. For the DNFs $A(\pi^p)$ and $B(\pi^p)$, $p \ge 2$, we take as component DNFs respectively

$$A(\pi_1^{p-1}),\ A(\overline{\pi_1^{p-1}}),\ A(\pi_2^{p-1}),\ A(\overline{\pi_2^{p-1}})$$

and

$$B(\pi_1^{p-1}),\ B(\overline{\pi_1^{p-1}}),\ B(\pi_2^{p-1}),\ B(\overline{\pi_2^{p-1}}),$$

where the functions π_1^{p-1} and π_2^{p-1} are the component functions of π^p. The irredundancy of the DNFs $A(\pi^p)$ and $B(\pi^p)$ follows from Lemma 1.

Theorem 1.

$$Y(n) \geq \frac{2^{\,n-\gamma_n}\sqrt{n+2}}{n}\left(1 - O(1/\sqrt{n})\right), \qquad \text{where } 2 \leq \gamma_n =$$

$$= 2^{-\varepsilon(\sqrt{n})} + 2^{\varepsilon(\sqrt{n})} \leq 5/2, \qquad \varepsilon(\sqrt{n}) = \log\sqrt{n} - [\log\sqrt{n}].$$

Proof. We compute $I(A(\pi^p))$, $I(B(\pi^p))$, and $Y(\pi^p)$ for a function π^p of n (essential) variables. We verify by induction.

by p, $1 \leq p \leq [\log(n+2)/3]$, that

$$I(B(\pi^p)) = 2^{n-2^p}, \qquad I(A(\pi^p)) \leq$$

$$\leq \varphi(p) \cdot 2^{\,n\cdot 2^{-p}+2p-3+2^{-p+1}},$$

where

$$\varphi(p) = 2^{-p}\prod_{k=1}^{p}\left(2^{-2^{-k}} + 2^{2^{-k}}\right).$$

Let π^{p+1} be a function of $n' = n_1 + n_2 + 2$ (essential) variables, where n_1 and n_2 are the numbers of (essential) variables in its components, with $0 \leq n_2 - n_1 \leq 1$. By (3),

$$I(B(\pi^{p+1})) = 2^{n'-2^{p+1}}.$$

By (2),

$$I(A(\pi^{p+1})) \leq \alpha(p)\cdot\varphi(p)\times$$

$$\times 2^{\frac{n_1+n_2}{2}\cdot 2^{-p}+2p-2+2^{-p+1}},$$

where $\alpha(p) = 2$ if $n_1 = n_2$; $\alpha(p) = 2^{-2^{-p-1}} + 2^{2^{-p-1}}$ if $n_1 = n_2 - 1$. Since $\frac{1}{2}\alpha(p)\varphi(p) = \varphi(p+1)$, we have

$$I(A(\pi^{p+1})) \leq \varphi(p+1)\cdot 2^{\,n'\cdot 2^{-p-1}+2(p+1)+3-2^{-p}}.$$

The verification for $p = 1$ is carried out from the same formulas (2) and (3).

Let us estimate $\varphi(p)$:

$$\varphi(p) = \frac{1}{2^p} \prod_{k=1}^{p} \left(1 + 2^{1/2^k - 1}\right) \Big/ \prod_{k=1}^{p} 2^{1/2^k} < 2.$$

Finally we have

$$I(A(\pi^p)) < 2^{\,n \cdot 2^{-p} + 2(p-1) + 2^{-p+1}}, \qquad I(B(\pi^p)) = 2^{n - 2^p},$$

$$Y(n) \geq Y(\pi^p) \geq I(B(\pi^p))/I(A(\pi^p)) \geq 2^{\,n - n \cdot 2^{-p} - 2^p - 2(p-1) - 3 \cdot 2^p}.$$

For

$$p = [\log \sqrt{n}] = \log \sqrt{n} - \varepsilon(\sqrt{n}), \qquad \varepsilon(\sqrt{n}) = \log \sqrt{n} - [\log \sqrt{n}],$$

we obtain the asserted estimate.

5. **Theorem 2.** $Y(n) \leq 2^n/n$.

6. Let us call the dimension of a function f, and denote by $\operatorname{Dim} f$, the number

$$\max(\operatorname{Dim} K),$$

where the maximum is taken over all conjunctions K absorbed by the function f. Denote by $\mathfrak{M}_{D(n)}$ the class of functions $f(x^n)$ for which $\operatorname{Dim} f \leq D(n)$. We shall say that a function $f(x^n)$ defines the class $\mathfrak{M}_{D(n)}$ if $\operatorname{Dim} f = D(n)$. If $D_1(n) < D_2(n)$, then

$$\mathfrak{M}_{D_1(n)} \subset \mathfrak{M}_{D_2(n)}.$$

Let us call the dimension of a DNF Φ, and denote by $\operatorname{Dim} \Phi$, the number $\max_{K \in \Phi}(\operatorname{Dim} K)$.

Lemma 2. $Y(\mathfrak{M}_{D(n)}) \leq 2^{D(n)}$.

Lemma 3. Let $\pi^1 = \pi^1(x^s, y^s, z, w)$. Then

$$\operatorname{Dim} \pi^1 = \operatorname{Dim} A(\pi^1) = s, \qquad Y(\pi^1) = 2^{\operatorname{Dim} \pi^1 - 1}.$$

To prove the lemma it is enough to establish that the only minimal conjunctions with respect to π^1 are the conjunctions from the forms $A(\pi^1)$ and $B(\pi^1)$,

$$\pi^1 = L(x^s)\overline{zw} \vee \overline{L}(x^s)z\overline{w} \vee L(y^s)\overline{zw} \vee \overline{L}(y^s)zw.$$

Theorem 3. For any integer-valued increasing function $\mu(n)$,

$$Y(\mathfrak{M}_{D(n)}) \sim 2^{D(n)}$$

when

$$D(n) = n/2 - \mu(n).$$

To prove the theorem, in each class $\mathfrak{M}_{D(n)}$, $D(n) = n/2 - \mu(n)$, it is necessary to specify such a defining (see the definition) function $\psi_n(u^n)$ that

$$Y(\psi_n) \cdot 2^{-\operatorname{Dim}\psi_n} \to 1$$

as $n \to \infty$. The functions $\pi^p(u^n)$ in the classes defined by them $\mathfrak{M}_{D(n)}$, $D(n) = \operatorname{Dim}\pi^p$, do not solve this problem, since one can show that, for $p \geq 0$,

$$Y(\pi^p) \cdot 2^{-\operatorname{Dim}\pi^p} \leq 2^{-2^p+1}.$$

But, as it turns out, having at our disposal the function $\pi^1(v^k)$, $k = 2s+2$, one can construct a function $\psi_n(u^n)$, $n \geq k+2$, for which

$$Y(\psi_n) \geq \frac{n-k}{n-k+4} \cdot 2Y(\pi^1(v^k)), \qquad \operatorname{Dim}\psi_n = \operatorname{Dim}\pi^1,$$

so that

$$Y(\psi_n) \cdot 2^{-\operatorname{Dim}\psi_n} \geq \frac{n-k}{n-k+4}. \tag{4}$$

(since $Y(\pi^1) \cdot 2^{-\operatorname{Dim}\pi^1} = \frac{1}{2}$).

Relying on the existence of the functions ψ_n, it is not difficult to show the validity of the theorem. Indeed, take $k = n - 2\mu + 2$. Then the function ψ_n will be defining for the class $\mathfrak{M}_{D(n)}$ with $D(n) = n/2 - \mu$, since

$\operatorname{Dim}\psi_n = \operatorname{Dim}\pi^1 = k/2 - 1 = n/2 - \mu = D(n)$. By virtue of (6), for this value of the parameter we shall have

$$Y(\psi_n) \cdot 2^{-\operatorname{Dim}\psi_n} \geq \frac{\mu(n)-1}{\mu(n)+1}.$$

Since $\mu(n) \to \infty$ as $n \to \infty$,

$$\lim Y(\psi_n) \cdot 2^{-\operatorname{Dim}\psi_n} \geq 1.$$

This fact, together with Lemma 2, proves the theorem.

Construction of the function $\psi_n(u^n)$. Divide the variables u^n into 4 groups: x^s, y^s, z^p, w^p, where $n = 2s + 2p,\ 2s + 2 = k$. For each $\sigma = (\sigma_1, \dots, \sigma_p)$ and $i = 1, \dots, p$, construct the conjunction

$$R_{i,\sigma} = \bigwedge_{\substack{j=1,\\ j\neq i}}^{p} z_j^{\sigma_j} w_j^{\sigma_j}.$$

Define the function

$$\psi_n(x^s, y^s, z^p, w^p) = \bigvee_{i,\sigma} R_{i,\sigma} \pi^1_{i,\sigma}(x^s, y^s, z_i, w_i),$$

where

$$\pi^1_{i,\sigma} = \pi^1(x^s, y^s, z_i^{\lambda_{i,\sigma}}, w_i^{\lambda_{i,\sigma}}), \qquad \lambda_{i,\sigma} = \sigma_1 + \dots + \sigma_p + \sigma_i \pmod 2.$$

Corollary. Let $D(n) = [\log n] + 1$. Then, by Theorem 3,

$$Y(\mathfrak{M}_{D(n)}) \sim n \cdot 2^{1-\varepsilon(n)}, \qquad \text{where } \varepsilon(n) = \log n - [\log n].$$

By direct counting it is not difficult to verify that the class $\mathfrak{M}_{D(n)}$, $D(n) = [\log n] + 1$, contains "almost all" functions of n variables.*

7. Let $J(T)$ be the number of letters in an irredundant D.N.F. T; $Z(f) = \max_{T_1,T_2}[J(T_1)/J(T_2)]$, where the maximum is taken over all pairs T_1, T_2 of irredundant D.N.F.' s of the function f; $Z(\mathfrak{M}) = \max_{f\in\mathfrak{M}} Z(f)$, where $\mathfrak{M}$ is a given set of functions f; $Z(n) = \max_{f(x^n)} Z(f(x^n))$.

Let T_1 and T_2 be irredundant D.N.F.' s of the function f.

$$\frac{J(T_1)}{J(T_2)} \cdot a(T_1, T_2) \le \frac{J(T_1)}{J(T_2)} \le \frac{J(T_1)}{J(T_2)} \cdot b(T_1, T_2),$$

where

$$a(T_1, T_2) = \min[r(K)/r(Q)], \qquad b(T_1, T_2) = \max[r(K)/r(Q)]$$

over all $K \in T_1,\ Q \in T_2$; K and Q are conjunctions from T_1 and T_2, respectively.

Applying this relation, one can obtain from Theorems 1, 2, 3 Theorems 1′, 2′, 3′, which are analogues of the first ones. For this, in each case one must estimate the factors a and b.

Theorem 1′.

$$Z(n) \ge \frac{2^{\,n-\gamma_n\sqrt{n}}}{\sqrt{n}}\,(1 - o(1)).$$

Theorem 2′.

$$Z(n) \le 2^n.$$

Theorem 3′. For any integer-valued increasing function $\mu(n)$,

$$Z(\mathfrak{M}_{D(n)}) \sim c(n) \cdot 2^{D(n)}, \qquad \text{where } c(n) = \frac{n}{[\, n - D(n)\,]}, \quad D(n) = n/2 - \mu(n).$$

Corollary.

$$Z(\mathfrak{M}_{[\log n]+1}) \sim n \cdot 2^{1-\varepsilon(n)}, \qquad \varepsilon(n) = \log n - [\log n],$$

i.e., by the corollary to Theorem 3,

$$Z(\mathfrak{M}_{[\log n]+1}) \sim Y(\mathfrak{M}_{[\log n]+1}).$$

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* This fact was communicated orally to the author by Yu. I. Zhuravlev.

Note: Figure translations are in progress. See original paper for figures.