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Abstract

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ON PROPOSITIONAL CALCULI WITH AN ADDITIONAL OPERATION

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Let T be a constructive propositional calculus. By adding to the axioms of T new axioms, each of which is a formula of T, and preserving the same rules of inference, we obtain a propositional calculus, which we shall denote by G. Let us add to the logical operations $\wedge, \vee, \supset, \neg$ an additional logical operation φ of one variable and extend the definition of a formula accordingly. Let us add to the axioms of the calculus G new axioms, each of which is a formula in the extended sense, and preserve in the extended calculus the rules of inference of the calculus T. The resulting calculus we shall call a propositional calculus with an additional operation and denote it by G^φ. A formula $\mathfrak{x}$ containing no occurrence of the operation φ will be called a formula of pure logic. We define the consistency* and completeness of the calculus G^φ. The calculus G^φ is consistent if every formula of pure logic provable in it is also provable in the calculus G. The inconsistency of the calculus G^φ is defined correspondingly. The calculus G^φ is complete if adding to its axioms a formula $\mathfrak{x}$ not provable in it as a new axiom makes the extended calculus inconsistent.

An additional operation φ is called single-valued if among the axioms describing it there is the axiom of single-valuedness

$$(x \sim y) \supset (\varphi(x) \sim \varphi(y)).$$

All calculi G^φ considered in this article contain the axiom of single-valuedness. In § 3 of the work ([1]) the calculus H, which is a formalization of the first matrix of Jaśkowski J_1, is considered, and 15 complete calculi H_i^φ $(1 \leq i \leq 15)$ are constructed, each of which describes a single-valued additional operation φ of one variable adjoined to the calculus H. In connection with this the question arises: are the consistent calculi H^φ containing the axiom of single-valuedness completely described by the system of calculi H_i^φ $(1 \leq i \leq 15)$, or not? A positive answer to this question is given by Theorem 1 of the present article. The question of a complete description of single-valued additional operations of one variable adjoined to the calculus T remains open. Some progress on it is provided by Theorem 2.

Let us introduce the definition of a model. In § 2 of the work ([2]) the definition is given of a matrix $\mathfrak{M}$ as a set with a distinguished element $\beta_{\mathfrak{M}}$ and with binary functions $\supset_{\mathfrak{M}}, \wedge_{\mathfrak{M}}, \vee_{\mathfrak{M}}$ and a unary function $\neg_{\mathfrak{M}}$ defined on this set, taking values in the same set. We shall extend this definition here by adding to the functions one more additional func-

* Consistency of calculi is understood here in a sense different from the commonly accepted one.

of one variable $\varphi_{\mathfrak{M}}$. We shall abbreviate the full notation for the matrix $\mathfrak{M}$

$$\mathfrak{M} = [A_{\mathfrak{M}}, \{\beta_{\mathfrak{M}}\}; \supset_{\mathfrak{M}}, \wedge_{\mathfrak{M}}, \vee_{\mathfrak{M}}, \neg_{\mathfrak{M}}, \varphi_{\mathfrak{M}}]$$

by means of $[A_{\mathfrak{M}}, \{\beta_{\mathfrak{M}}\}]$. The independent variables of functions considered in matrices are denoted by α_i $(i = 1, 2, \dots, n, \dots)$.

Let us establish a one-to-one correspondence between the formulas of the calculus G^{φ} and the functions of some matrix $\mathfrak{M}$. To the formula $\mathfrak{X}(a_1, \dots, a_n)$ we put in correspondence the function $\mathfrak{X}'(\alpha_1, \dots, \alpha_n)$, obtained by replacing the variables of the formula $a_1, \dots, a_n$ by the independent variables $\alpha_1, \dots, \alpha_n$, and the logical operations $\supset, \wedge, \vee, \neg, \varphi$ by the corresponding functions of the matrix.

A function of the matrix $\mathfrak{M}$ will be called identical if it always assumes the distinguished value $\beta_{\mathfrak{M}}$.

Definition 1. The matrix $\mathfrak{M}$ is a **model of the calculus** G^{φ} if the function of the matrix $\mathfrak{X}'(\alpha_1, \dots, \alpha_n)$, corresponding to the formula $\mathfrak{X}(a_1, \dots, a_n)$ provable in the calculus G^{φ}, is identical in the matrix $\mathfrak{M}$.

The matrix $\mathfrak{M}$ is an **exact model of the calculus** G^{φ} if: 1) $\mathfrak{M}$ is a model of the calculus G^{φ}; 2) to every function $\mathfrak{X}'(\alpha_1, \dots, \alpha_n)$ identical in the matrix $\mathfrak{M}$ there corresponds a formula $\mathfrak{X}(a_1, \dots, a_n)$ provable in the calculus G^{φ}.

Definition 2. Let $\mathfrak{M} = [A_{\mathfrak{M}}, \{\beta_{\mathfrak{M}}\}]$ be a model of the calculus G^{φ}. Let $A_{\mathfrak{M}'}$ be such a subset of the set $A_{\mathfrak{M}}$ that the functions of the matrix $\mathfrak{M}$, defined on the sum of the sets $A_{\mathfrak{M}'} + \{\beta_{\mathfrak{M}}\}$, take their values from this same set. Obviously, then we obtain a matrix $\mathfrak{M}' = [A_{\mathfrak{M}'}, \{\beta_{\mathfrak{M}}\}]$ with the same functions as in the matrix $\mathfrak{M}$. The matrix $\mathfrak{M}'$ is called a **submodel** of the model $\mathfrak{M}$.

In the usual way one defines an isomorphism of two matrices $\mathfrak{M}_1, \mathfrak{M}_2$ (or models $\mathfrak{M}_1, \mathfrak{M}_2$). From Definitions 1 and 2 we obtain two lemmas.

Lemma 1. *If the matrix $\mathfrak{M}$ is a model of the calculus G^{φ}, and the matrix $\mathfrak{M}'$ is a submodel of the model $\mathfrak{M}$, then the matrix $\mathfrak{M}'$ is also a model of the calculus G^{φ}.*

Lemma 2. *Let the matrix $\mathfrak{M}_1$ be a model of the calculus G_1^{φ}, and the matrix $\mathfrak{M}_2$ a model of the calculus G_2^{φ}. If the matrices $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic, then $\mathfrak{M}_1$ is a model of the calculus G_2^{φ}, and $\mathfrak{M}_2$ is a model of the calculus G_1^{φ}.*

Let us construct an exact model of an arbitrary calculus G^φ, which we shall call the natural model. The set of formulas of the calculus G^φ is divided into classes of equivalent formulas (formulas $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are equivalent if in the calculus the formulas $\mathfrak{X}_1 \supset \mathfrak{X}_2$, $\mathfrak{X}_2 \supset \mathfrak{X}_1$ are provable). Denote the class of formulas equivalent to the formula $\mathfrak{X}$ by $\widetilde{\mathfrak{X}}$, and the aggregate of the classes of equivalent formulas of the calculus G^φ by $\widetilde{G^\varphi}$. We now define the matrix $\mathfrak{C}$. The set of elements of the matrix $\mathfrak{C}$ is the class $\widetilde{G^\varphi}$. The functions $\boxplus_{\mathfrak{C}}$* are defined as follows:

$$(\widetilde{\mathfrak{X}}_1 \boxplus_{\mathfrak{C}} \widetilde{\mathfrak{X}}_2) = (\widetilde{\mathfrak{X}_1 \boxplus \mathfrak{X}_2}),$$

and the matrix functions $\neg_{\mathfrak{C}}, \varphi_{\mathfrak{C}}$ are defined analogously. Obviously, the matrix $\mathfrak{C}$ is an exact model of the calculus G^φ with distinguished element $\tilde{1}$**.

The finite aggregate of formulas

$$\{\varphi(0), \varphi(1), \varphi(\varphi(0)), \varphi(\varphi(1)), \varphi(a \vee \neg a)\}$$

together with their negations and double negations will be called the basic set. A nonempty subset of the basic set of formulas will be called

* $\boxplus$ denotes a binary logical operation $\supset, \wedge, \vee$.

** 1 is the notation for the formula $a \supset a$, and 0 for the formula $a \wedge \neg a$.

by the choice of the case. Let $\{\mathfrak{X}_1, \dots, \mathfrak{X}_k\}$ be a finite class of formulas. Add to the axioms of the calculus G^φ the axioms $\mathfrak{X}_1, \dots, \mathfrak{X}_k$. The resulting calculus, which is an extension of the calculus G^φ, will be denoted by $G^\varphi[\mathfrak{X}_1, \dots, \mathfrak{X}_k]$.

Definition 3. A class of case choices $\{X_i\}$ $(1 \leq i \leq n)$ will be called **complete** if, whatever consistent calculus H^φ may be given, there is a case choice

$$X_{i_0} = \{\mathfrak{X}_1, \dots, \mathfrak{X}_p\} \quad (1 \leq i_0 \leq n)$$

such that the calculus

$$H^\varphi[\mathfrak{X}_1, \dots, \mathfrak{X}_p]$$

is consistent.

Lemma 3. Let $\mathfrak{X}$ be a formula of the basic set, and let H^φ be an arbitrary consistent calculus. Then one of the two calculi

$$H^\varphi[\neg\mathfrak{X}], \qquad H^\varphi[\neg\neg\mathfrak{X}]$$

is consistent.

With the aid of Lemma 3 and the uniqueness axiom, a complete class of case choices is constructed,

$$\{X_i\}\ (1 \le i \le 15), \tag{1}$$

whose elements are as follows:

$$
\begin{aligned}
X_1 &= \{\varphi(0), \varphi(1), \varphi(a \vee \neg a)\}; & X_2 &= \{\varphi(0), \varphi(1), \neg\neg\varphi(a \vee \neg a)\};\\
X_3 &= \{\varphi(0), \neg\neg\varphi(1), \varphi(\varphi(1))\}; & X_4 &= \{\varphi(0), \neg\neg\varphi(1), \varphi(\varphi(1))\};\\
X_5 &= \{\neg\varphi(0), \varphi(1), \varphi(a \vee \neg a)\}; & X_6 &= \{\neg\varphi(0), \varphi(1), \neg\neg\varphi(a \vee \neg a)\};\\
X_7 &= \{\neg\varphi(0), \neg\varphi(1), \neg\varphi(a \vee \neg a)\}; & X_8 &= \{\neg\varphi(0), \neg\neg\varphi(1), \varphi(\varphi(1))\};\\
X_9 &= \{\neg\varphi(0), \neg\neg\varphi(1), \neg\neg\varphi(\varphi(1))\}; & X_{10} &= \{\neg\neg\varphi(0), \varphi(1), \varphi(\varphi(0))\};\\
X_{11} &= \{\neg\neg\varphi(0), \neg\neg\varphi(1), \varphi(\varphi(0))\}; & X_{12} &= \{\neg\neg\varphi(0), \neg\varphi(1), \neg\varphi(\varphi(0))\};\\
X_{13} &= \{\neg\neg\varphi(0), \varphi(1), \neg\neg\varphi(\varphi(0))\}; & X_{14} &= \{\neg\neg\varphi(0), \neg\neg\varphi(1),\\
& & &\quad \neg\neg\varphi(\varphi(0)), \neg\neg\varphi(\varphi(1))\};\\
X_{15} &= \{\neg\varphi(0), \neg\varphi(1), \neg\varphi(a \vee \neg a)\}.
\end{aligned}
$$

The complete class of case choices (1) just constructed makes it possible to prove the following theorem:

Theorem 1. Let H^φ be an arbitrary consistent calculus. There exists a calculus H_i^φ $(1 \le i \le 15)$ such that all axioms of H^φ are derivable in the calculus H_i^φ.

We indicate the scheme of the proof of Theorem 1. Let H^φ be an arbitrary consistent calculus. Among the elements of the complete case choice (1), on the basis of Definition 3, there is a case choice

$$X_i = \{\mathfrak{X}_1, \dots, \mathfrak{X}_k\}$$

such that the calculus

$$H^\varphi[\mathfrak{X}_1, \dots, \mathfrak{X}_k]$$

is consistent. Let $\mathfrak{C}$ be its natural model. With the aid of the uniqueness axiom and the axioms $\mathfrak{X}_1, \dots, \mathfrak{X}_k$, a submodel $\mathfrak{C}'$ of the model $\mathfrak{C}$ is found, containing the three elements $\{\hat{0}, \hat{1}, \mathfrak{B}\}$, where $\mathfrak{B}$ is one of the formulas

$$\varphi(0),\ \varphi(1),\ \varphi(a \vee \neg a),\ a \vee \neg a.$$

Each of the calculi H_i^φ $(1 \le i \le 15)$ has an exact model of three elements $\{\lambda, \mu, \sigma\}$. The model $\mathfrak{C}'$ turns out to be isomorphic to an exact model of one of the calculi H_i^φ. Consequently, on the basis of Lemma 2, the calculus

$$H^\varphi[\mathfrak{X}_1, \dots, \mathfrak{X}_k]$$

has as a model the exact model of some calculus H_i^φ. By virtue of the completeness of the calculus H_i^φ, all axioms of the calculus H^φ are derivable in the calculus H_i^φ.

Corollary. If the calculus H^{φ} is consistent and complete, then there exists a calculus H_i^{φ} such that the classes of provable formulas of the calculi H^{φ}, H_i^{φ} coincide.

Theorem 2. Let T^{φ} be an arbitrary consistent constructive propositional calculus with an additional operation. There will be found–

there is a calculus H_i^{φ} ($1 \leqslant i \leqslant 15$) such that all axioms of the calculus T^{φ} are derivable in the calculus H_i^{φ}.

For the proof of Theorem 2 the notion is used of the consistency of the calculus T^{φ} relative to the calculus H. The calculus T^{φ} is consistent relative to the calculus H if every formula of pure logic that is derivable in the calculus T^{φ} is also derivable in the calculus H.

The calculus T^{φ} is extended, by means of a certain selection of cases, to the calculus $T^{\varphi}[\mathfrak{X}_1, \dots, \mathfrak{X}_q]$, consistent relative to H, and then an argument analogous to that given in Theorem 1 is carried out.

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REFERENCES

1. Ya. S. Smetanich, *Tr. Mosk. matem. obshch.*, **9**, 357 (1960).

2. B. Yu. Pilchak, *Ukr. matem. zhurn.*, **4**, 174 (1952).

Note: Figure translations are in progress. See original paper for figures.