On Locally Extremal Groups and Groups with the Π-Minimality Condition

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Abstract

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On Locally Extremal Groups and Groups with the Π-Minimality Condition

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In the present article necessary and sufficient conditions are given under which a locally finite group, every countable quasifull subgroup of which (i.e., a subgroup having no proper subgroups of finite index) is solvable or locally nilpotent (in particular, a solvable group), satisfies the Π-minimality condition* (Theorems 1 and 2).

It is known (see ([1])) that, under certain sufficiently general restrictions, the class of groups with the condition of primary minimality (i.e., p-minimality for each p) coincides with the class of layerwise extremal groups, i.e., groups in which every set of elements of one and the same order generates an extremal group. **In the paper ([1]) it was noted that many properties of layerwise extremal groups are analogous to the corresponding properties of layerwise finite groups. As is known ([3]), layerwise finite groups can be characterized as locally normal groups with special (in the sense of S. N. Chernikov) Sylow p-subgroups. The same role (see Theorem 3) with respect to layerwise extremal groups is played by locally extremal groups (we shall call a group** locally extremal** if every element of it is contained in some extremal normal divisor of it). In the present article certain properties of locally extremal groups are formulated; in particular, a condition is given for the embeddability of countable locally extremal groups in a direct product of extremal groups (Theorem 6).

1. **Theorem 1.** *A locally finite group, every countable quasifull subgroup of which is solvable or locally nilpotent, satisfies the* Π*-minimality condition if and only if the set of its* Π*-elements generates an extremal subgroup.*

Corollary 1. *A periodic solvable group satisfies the* Π*-minimality condition if and only if the set of its* π*-elements generates an extremal subgroup.*

Corollary 2. *A quasifull solvable group with the* Π*-minimality condition decomposes into the direct product of a complete Abelian* Π*-group with the minimality condition and a periodic group containing no* Π*-elements.*

Definition. We shall call the Π**-center** of an arbitrary group (Π is some set of prime numbers) the intersection of the centralizers of all its Π-elements.

It is obvious that if, in some locally finite group $\mathfrak{G}$, the set of its Π-elements generates an extremal subgroup (which occurs, in particular, for groups satisfying the conditions of Theorem 1), then all Sylow Π-subgroups of the group $\mathfrak{G}$ and its factor group by the Π-center are extremal. It turns out that the converse proposition is also true.

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* For the definition see the author' s paper ([1]).

** In accordance with the paper ([2]), by an extremal group we understand a finite group or a finite extension of an Abelian group with the minimality condition.

Theorem 2. The set of Π-elements of a locally finite group $\mathfrak{G}$ generates an extremal subgroup if and only if all Sylow Π-subgroups of the group $\mathfrak{G}$ and its factor group by the Π-center are extremal.

Corollary. A locally finite group is layerwise extremal if and only if all its Sylow p-subgroups are special and the factor groups by each p-center are extremal.

In the proof of Theorem 2 the following propositions are used, which are also of some independent interest.

Lemma 1. A periodic group of automorphisms of an extremal group is extremal.

If the extremal group here is complete, then its periodic group of automorphisms is even finite ([4]).

Lemma 2. The commutator subgroup of a central extension of an arbitrary group by means of an extremal (layerwise extremal) group is extremal (layerwise extremal).

In the case of a finite central extension, its commutator subgroup is even finite ([5]).

Lemma 3. A central extension of a periodic group by means of an extremal group decomposes into the product of its center and some extremal group.

The corollary to Theorem 2 shows that layerwise extremal groups are characterized by the fact that their Sylow p-subgroups are special and the factor groups by each p-center are extremal. Groups satisfying the first of these conditions were considered in ([2]). Naturally there arises the question of the structure of groups satisfying the second condition. The next section is devoted to the solution of this question.

2. **Theorem 3.** The class of layerwise extremal groups coincides with the class of locally extremal groups with special Sylow p-subgroups.

Corollary. For locally extremal groups the conditions of primary minimality and specialness of the Sylow p-subgroups for all p are equivalent.

In the general case, as the example from ([1]) shows, these conditions are not equivalent, even for soluble groups.

Theorem 4. A central extension of a periodic group by means of a layerwise extremal group is a locally extremal group.

An analogous assertion for central extensions by means of layerwise finite groups was proved in ([6]).

Corollary. A central extension of a layerwise extremal group by means of a layerwise extremal group is a layerwise extremal group.

Lemma 4. In a locally extremal group every extremal subgroup is contained in some extremal normal divisor of it.

We can now answer the question posed at the end of Section 1.

Theorem 5. The factor groups of a locally finite group $\mathfrak{G}$ by each p-center are extremal if and only if the group $\mathfrak{G}$ is a central extension of a periodic group by means of a layerwise extremal group.

3. In Hall' s paper ([7]) some connections are established between countable locally normal groups and countable direct products of finite groups. It turns out that analogous connections exist between countable locally extremal groups and countable direct products of extremal groups. The proofs of the corresponding propositions are based on the same ideas as in Hall; in addition, Lemmas 1 and 5 are used.

Lemma 5. If $\mathfrak{G}$ is an arbitrary extremal subgroup of a group $\mathfrak{A}$, embeddable in a complete direct product of extremal groups, then in the group $\mathfrak{A}$

there exists in it an invariant subgroup $\mathfrak{N}$ such that $\mathfrak{G} \cap \mathfrak{N} = 1$ and the factor group $\mathfrak{A}/\mathfrak{N}$ is extremal.

Theorem 6. A countable locally extremal group can be embedded in a direct product of extremal groups if and only if it can be embedded in a complete direct product of extremal groups.

Corollary 1. The factor group of an arbitrary countable locally extremal group by its center can be embedded in a direct product of extremal groups.

Corollary 2. A countable locally extremal group without center can be embedded in a direct product of extremal groups.

Theorem 7. Every countable locally extremal group is isomorphic to a factor group of some subgroup of a countable direct product of extremal groups.

In view of Lemma 1, this theorem follows directly from Theorem 3.1 of the paper ([7]).

Lemma 6. If $\mathfrak{N}$ is a certain normal divisor of a layer-extremal group $\mathfrak{G}$, for which the factor group $\mathfrak{G}/\mathfrak{N}$ is extremal, then in the group $\mathfrak{G}$ there exists an extremal normal divisor $\mathfrak{P}$ such that $\mathfrak{G} = \mathfrak{N}\mathfrak{P}$.

Theorem 8. Every layer-extremal group $\mathfrak{G}$ is representable as the product of two of its normal divisors, each of which is a direct product of extremal groups invariant in $\mathfrak{G}$.

In view of Lemma 6, the proof of Theorem 8 is carried out in the same way as the proof of Theorem 2.4 in the paper ([7]).

Remark to Theorem 8. As is known ([1]), every layer-extremal group is countable and is a subgroup of some direct product of extremal groups. The author does not know whether the assertion of Theorem 8 is true for arbitrary subgroups of countable direct products of extremal groups.

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Note: Figure translations are in progress. See original paper for figures.