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1961

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Abstract

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ON ONE PROPERTY OF SOLUTIONS OF A HYPOELLIPTIC EQUATION

(Presented by Academician P. S. Aleksandrov, 25 X 1960)

Let $u(x)$ be a solution of the partial differential equation with constant coefficients

$$P\left(i\frac{\partial}{\partial x}\right)u(x) = 0. \tag{1}$$

In the present note we consider the question of the relation between the growth of the function $u(x)$ at infinity and its smoothness. In § 1 two theorems are proved which show what conditions must be imposed on the polynomial $P(s)$ in order that every solution $u(x)$ of equation (1), defined for all x, under one or another restriction on growth at infinity, be infinitely differentiable. In § 2 the smoothness of solutions of a hypoelliptic equation is discussed. Hypoelliptic equations were introduced by L. Hörmander in (^{1}), where it was proved that, if $u(x)$ is a solution of such an equation, then $u(x)$ is infinitely differentiable, and if W is a bounded domain, then

$$\max_{x\in W}\left|D^b u(x)\right| \le c^b\Gamma\left(\frac{k}{\gamma}\right),$$

where $\Gamma(z)$ is Euler' s function, and γ is the genus of the hypoelliptic equation. In § 2 it is shown that, under certain restrictions on the growth of $u(x)$ at infinity, this estimate of the smoothness of the function $u(x)$ can be substantially improved.

1. We first impose on $u(x)$ the strongest condition. Let $u(x)$, as $x \to \infty$, grow no faster than a polynomial. Under these assumptions it is proved in (^{2}), p. 165, that if all real solutions of the algebraic equation

$$P(\sigma) = 0 \tag{2}$$

lie in a bounded part of the plane, then $u(x)$ is an entire analytic function of order of growth not higher than one. We formulate an assertion which, in a certain sense, is the converse of this.

Theorem 1. *If every continuous bounded solution of equation (1) has continuous first-order derivatives, then all real solutions of equation (2) lie in a bounded part of the plane.*

Proof. Consider the normed space B of continuous bounded functions, defined for all x, with norm

$$\|\varphi(x)\|_B = \sup |\varphi(x)|,$$

and the normed space D of continuous functions, defined on the whole real plane and having continuous first derivatives in the circle $|x| \leq 1$. Define the norm in the space D by

$$\|\varphi(x)\|_D = \sup |\varphi(x)| + \sum_{i=1}^{n} \max_{|x|\leq 1} \left| \frac{\partial \varphi(x)}{\partial x_i} \right|.$$

All functions from $B(D)$ that are solutions of equation (1) (in the generalized sense) form in the normed space $B(D)$ a linear subspace,

which is, obviously, closed. Denote this subspace by $\widetilde{B}(\widetilde{D})$. It is clear that $\widetilde{B}(\widetilde{D})$ itself is a complete normed space. Moreover, from the conditions of the theorem it follows that each function in $\widetilde{B}$ has continuous first derivatives and, consequently, belongs to $\widetilde{D}$. Thus we have a natural continuous mapping of the space $\widetilde{D}$ onto the whole space $\widetilde{B}$. Applying now the well-known Banach theorem, we obtain that the inverse mapping is continuous, and, consequently, there exists a constant $C > 0$ such that

$$\|u(x)\|_D \leq C\|(ux)\|_B$$

for every $u(x) \in \widetilde{B}$. Let σ be a real root of equation (2). It is easy to verify that $u(x) = e^{-i\langle x\sigma\rangle}$ is, in this case, a solution of equation (1). Since

$$\|e^{-i\langle x\sigma\rangle}\|_B = 1 \quad \text{and} \quad \|e^{-i\langle x\sigma\rangle}\|_D = 1 + \sum_{i=1}^{n} |\sigma_i|,$$

we obtain

$$1 + \sum_{i=1}^{n} |\sigma_i| \leq C.$$

Thus the theorem is proved.

Lemma. If every continuous solution $u(x)$ of equation (1) such that

$$|u(x)| \leq e^{a|x|}, \tag{3}$$

is a continuously differentiable function, then there exists a constant $C > 0$ such that from $P(s) = 0$, $s = \sigma + i\tau$, $|\tau| \leq a$, it follows that $|\sigma| \leq C$.

The last condition means that the strip $|\tau| \leq a$ cuts out from the variety of all complex roots of the equation

$$P(s) = 0, \qquad s = \sigma + i\tau, \tag{4}$$

a bounded set. The proof of this lemma can be obtained by an almost verbatim repetition of the preceding one, if as the spaces B and D one takes the spaces of continuous functions with the norms

$$\|\varphi\|_B = \sup|\varphi(x)|e^{-a|x|}, \qquad \|\varphi(x)\|_D = \sup|\varphi(x)|e^{-a|x|} + \sum_{i=1}^{n} \max_{|x|\leq 1} \left|\frac{\partial\varphi(x)}{\partial x_i}\right|.$$

A direct consequence of the lemma is the following theorem:

Theorem 2. If every continuous solution of equation (1) which, for some $a > 0$, satisfies inequality (3) is a continuously differentiable function, then equation (1) is hypoelliptic.

Recall that equation (1) is called hypoelliptic if, for the roots of equation (4), from $|\sigma| \to \infty$ it follows that also $|\tau| \to \infty$. Thus, for nonhypoelliptic equations no information about the smoothness of the solution $u(x)$ can be obtained from estimate (3).

2. Suppose now that (1) is a hypoelliptic equation of type γ. This means that the variety of roots of equation (4) lies in the region

$$|\tau| \geq c|\sigma|^{\gamma} - c_1.$$

Theorem 3. If $u(x)$ is a solution of the hypoelliptic equation (1) and

$$|u(x)| \leq Ce^{a|x|^{1/\beta}}, \qquad 0 < \beta \leq 1, \tag{5}$$

then in any bounded domain

$$|D^k u(x)| \leq C^k \Gamma\left[\frac{(1-\beta)}{\gamma}\, k\right]$$

with some constant $C > 0$.

Proof. As shown in (3), there exists a constant $M > 0$ such that for $l \leq p$, where p is the order of equation (1),

$$|D^l u(x)| \leq M \max_{|\xi|\leq 1} |u(x+\xi)|.$$

From this inequality and estimate (5) we obtain that

$$|D^l u(x)| \leq Ce^{a|x|^{1/\beta}}, \qquad 0 \leq l \leq p, \tag{6}$$

with other constants C and a.

Let $\alpha(x)$ be an infinitely differentiable function such that $\alpha(x)=1$ for $|x|\leqslant 1$ and $\alpha(x)=0$ for $|x|\geqslant 2$. Consider the function

$$f_k(x)=P(D)u(x)\alpha(k^{-\beta}x).$$

It is clear that $f_k(x)$ is different from zero only when

$$k^\beta<|x|<2k^\beta.$$

Moreover, from (6) it follows that $|f_k(x)|\leqslant A^k$. Let us now consider a fundamental solution $\mathcal{E}(x)$ of equation (1). Since $P(D)\mathcal{E}(x)=\delta(x)$, for $|x|<k^\beta$

$$D^k u(x)=P(D)u(x)\alpha(k^{-\beta}x)*D^k\mathcal{E}(x)=f_k(x)*D^k\mathcal{E}(x),$$

$$|D^k u(x)|=\left|\int_{k^\beta<|\xi|<2k^\beta} f_k(\xi)D^k\mathcal{E}(x-\xi)\,d\xi\right|\leqslant Ck^{n\beta}A^k\max_{k^\beta\leqslant|\xi|\leqslant 2k^\beta}|D^k\mathcal{E}(x-\xi)|. \tag{7}$$

If $x\in W$ lies in a bounded domain, then for sufficiently large k

$$\max_{k^\beta\leqslant|\xi|\leqslant 2k^\beta}|D^k\mathcal{E}(x-\xi)|\leqslant\max_{0.5k^\beta\leqslant|y|\leqslant 3k^\beta}|D^k\mathcal{E}(y)|. \tag{8}$$

Thus, in order to obtain an estimate for the growth of the derivatives $D^k u(x)$, $x\in W$, it is necessary to estimate $D^k\mathcal{E}(x)$. As was shown in (4), $\mathcal{E}(x)$ can be represented in the form

$$\mathcal{E}(x)=G(x)+F(x), \tag{9}$$

where $G(x)$ is an entire analytic function of order of growth not exceeding one, and

$$|D^kF(x)|\leqslant C^ke^{l|x|}\int_1^\infty e^{-L|x|r^\gamma}r^{k+n-1}\,dr\leqslant C^ke^{b|x|}\int_0^\infty e^{-L|x|r^\gamma}r^{k+n-1}\,dr. \tag{10}$$

Making the change of variable $L|x|r^\gamma=y$, $dr=\dfrac{1}{\gamma}(L|x|)^{-1/\gamma}y^{1/\gamma-1}\,dy$, we obtain

$$|D^kF(x)|\leqslant C_1^ke^{b_1|x|}|x|^{-k/\gamma}\int_0^\infty e^{-y}y^{\frac{k+n}{\gamma}-1}\,dy=C_1^ke^{b_1|x|}|x|^{-k/\gamma}\Gamma\left(\frac{k+n}{\gamma}\right).$$

Applying Stirling’ s formula, we shall have, for $|x|>1$,

$$|D^kF(x)|\leqslant C_2^{k+|x|}\left(\frac{k}{|x|}\right)^{k/\gamma}. \tag{11}$$

From (11) we obtain that

$$\max_{0.5k^\beta\leqslant|x|\leqslant 3k^\beta}|D^kF(x)|\leqslant C_2^{4k}(2k)^{\frac{1-\beta}{\gamma}k}\leqslant C_3^k\Gamma\left[\frac{(1-\beta)k}{\gamma}\right]. \tag{12}$$

Let us now estimate $D^{kG}(x)$ for $|x| < 3k$. Since $G(x)$ is an entire function of order of growth not exceeding one, $|G(z)| \leqslant Ce^{a|z|}$. Applying Cauchy' s formula with respect to one of the variables,

$$D^{kG}(x) = \frac{k!}{2\pi i} \int_L \frac{G(z)}{(z_1 - x_1)^{k+1}}\, dz_1,$$

where as the contour L we take the circle $|z_1| = 4k$. In this case we obtain that for $|x| < 3k$

$$|D^{kG}(x)| \leqslant \frac{4Ck!}{k^k} e^{b_2 k} \leqslant B^k. \tag{13}$$

Relations (7), (8), (9), (12), and (13) prove the theorem.

Let us derive several consequences from this theorem.

Corollary 1. If $u(x)$ satisfies the conditions of the preceding theorem and $\beta \geqslant 1 - \gamma$, then $u(x)$ is an analytic function of order of growth not exceeding

$$\frac{\gamma}{\gamma + \beta - 1}.$$

Corollary 2. If $u(x)$ is a solution of a hypoelliptic equation and satisfies inequality (3), then $u(x)$ is an entire analytic function of order of growth not exceeding one.

Corollary 3. If $u(x)$ is a solution of an elliptic equation that satisfies inequality (5), then $u(x)$ is an entire function whose order of growth does not exceed $1/\beta$.

This assertion follows from Corollary 1 if one notes that for elliptic equations $\gamma = 1$. In particular, for the Laplace equation we obtain that any harmonic function $u(x)$ which at infinity grows no faster than $Ce^{a|x|^\alpha}$ is an entire analytic function of several complex variables which in the complex plane satisfies the inequality $|u(z)| \le C_1 e^{a_1 |z|^{\alpha_1}}$, if $\alpha \ge 1$.

Corollary 4. If the hypoelliptic equation (1) has genus γ, then there exists no solution $u(x)$ of equation (1) which vanishes in some domain and satisfies the inequality

$$|u(x)| \le Ce^{a|x|^{\frac{1}{1-\gamma}}}.$$

As an example let us consider the heat equation

$$\frac{\partial u(x_1, x_2)}{\partial x_1} = \frac{\partial^2 u(x_1, x_2)}{\partial x_2^2}.$$

It is not hard to compute that the genus of this equation is equal to $1/2$. Thus, every solution $u(x)$ of the heat equation which grows at infinity no faster than

$$Ce^{a(x_1^2 + x_2^2)}$$

is an analytic function in both variables.

3. An infinitely differentiable function $u(x)$, defined in a domain W, is called a function of class ρ in the direction y, if for every compact set $K \subset W$ there exists a constant C such that

$$\max_{x \in K} \left| \langle yD \rangle^k u(x) \right| \leq C^k \Gamma(\rho k)$$

(see ([1], p. 106). It is well known that all solutions of elliptic equations are analytic and, consequently, are functions of class 1 in all directions. We shall formulate an assertion which in a certain sense is converse to this.

Theorem 4. If every infinitely differentiable solution of equation (1) is a function of class 1 in the direction y, then the vector y is orthogonal to all real solutions of the equation $P_0(\xi) = 0$, where $P_0(s)$ is the principal part of the polynomial $P(s)$.

The proof of this theorem follows from the fact that for any characteristic ξ, $P_0(\xi) = 0$, there exists an infinitely differentiable solution $u(x)$ which is identically zero in the half-space $\langle x, \xi \rangle \geq 0$ (([1], p. 86). Such a solution $u(x)$ can be a function of class 1 only for those y for which $\langle y, \xi \rangle = 0$.

Theorem 5. Suppose that in some domain W every infinitely differentiable solution of equation (1) is a function of class ρ in the direction y. If y is not orthogonal to any real solution of the equation $P_0(\xi) = 0$, where $P_0(s)$ is the principal part of the polynomial $P(s)$, then $P(s)$ is a hypoelliptic polynomial and every solution of equation (1) is a function of class ρ in any other direction.

In conclusion I express my gratitude to Prof. G. E. Shilov for his great attention to this work and for a number of valuable comments.

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Received
11 X 1960

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Note: Figure translations are in progress. See original paper for figures.

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