Academician of the Academy of Sciences of the Azerbaijan SSR Z. I. KHALILOV

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

Academician of the Academy of Sciences of the Azerbaijan SSR Z. I. KHALILOV

ON THE STABILITY OF SOLUTIONS OF AN EQUATION IN A BANACH SPACE*

Let B be a Banach space. Consider the collection of functions $x(t)$, defined on the half-line $J = [0, +\infty)$, with values in B. By L we denote the collection of functions $x(t)$ that are locally Bochner integrable and have a countable family of compatible norms

$$\|x\|_n = \int_0^n \|x(t)\| \, dt, \qquad n = 1, 2, \ldots .$$

Let $U(t, s)$, for any fixed values of t and s $(0 \le s \le t < +\infty)$, be a linear bounded operator acting in the space B and satisfying the conditions:

1°. $U(t, s)$ is strongly continuous in the aggregate of t and s; $U(t, t) = I$ (I is the identity operator).

2°. $U(t, s)U(s, 0) = U(t, 0)$, $0 \le s \le t$.

Consider the nonlinear integral equation

$$x(t) = U(t, 0)x_0 + \int_0^t U(t, s)h(x(s), s) \, ds, \qquad x_0 \in B. \tag{1}$$

Obviously, $x_0 = x(0)$. We shall establish conditions under which the solutions of equation (1) are stable.

By M_1 we denote the set of functions $x(t)$ having the representation

$$x(t) = U(t, 0)x_0, \qquad x_0 \in B. \tag{2}$$

Obviously, the functions belonging to M_1 are continuous on J.

By M_2 we denote the set of functions $x(t)$ having the representation

$$x(t) = U(t, 0)x_0 + \int_0^t U(t, s)f(s) \, ds, \qquad x_0 \in B, \tag{3}$$

where $f(t) \in \mathbf{B}$; $\mathbf{B}$ is a Banach space stronger than L, whose norm we denote by $|f|_{\mathbf{B}}$. For example, C is the set of continuous functions with norm $|x| = \sup_{t \in J} \|x(t)\|$. It is not difficult to verify that the functions of the set M_2 are also continuous on J.

Lemma 1. *The set X of all bounded functions from the set M_1 is a subspace of the space C.*

Denote by B_0 the collection of all elements of B that are initial values of bounded functions in M_1, $|x| < +\infty$. As examples show, B_0, generally speaking, is not closed.

* The contents of the present note were reported at the Fifth All-Union Conference on Functional Analysis in Baku in October 1959.

Lemma 2. If B_0 is closed, then there exists a positive number S such that, for all $x(t) \in X$,

$$|x| \leq S\|x(0)\|,$$

where $|x|$, here and below, denotes the norm in C.

Proof is carried out by applying the theorem on the continuity of the inverse of a one-to-one linear operator to the operator T, which assigns to each element of X its initial value x_0.

Let there exist a set B_1, a complement of the closed B_0. It is not always closed. Introduce the projection operators P_0, P_1, $x_0 = P_0 x$, $x_1 = P_1 x$, $x \in B$, $x_0 \in B_0$, $x_1 \in B_1$.

Lemma 3. Suppose B_0 is closed and has a closed complement B_1. Let there correspond to each $f(t) \in B$ at least one bounded function from M_2. Then there exists a constant $K > 0$ such that to each $f(t) \in B$ there corresponds an $x(t) \in M_2$ satisfying the inequality $|x| \leq K|f|_B$; this function can be chosen so that $x(0) \in B_1$, and then it is determined uniquely.

Proof. Let Y be the set of all functions $x(t)$ satisfying the conditions: $x(0) \in B_1$, $x(t)$ is continuous and bounded, and

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s)\,ds, \qquad f(t) \in B.$$

In this linear manifold Y introduce the norm

$$|x|_Y = |x| + |f|_B.$$

It is easy to prove that Y, so normed, is a Banach space.

Let T be the linear mapping of Y into B: $Tx = f$. From the inequality

$$|Tx|_B = |f|_B \leq |x| + |f|_B = |x|_Y$$

it follows that T is bounded and $\|T\| \leq 1$. It is not difficult to show that there exists T^{-1}, $D(T^{-1}) = B$, if one takes $x_1 \in B_1$ for $x(0)$.

We now apply to T the theorem on the continuity of the inverse of a one-to-one linear transformation. Then $\|T^{-1}\|$ exists and $|x_1| \leq k|f|_B$, where $K = \|T^{-1}\|{-1}$.

In what follows we shall assume that the nonlinear operator $h(x(t), t)$ is such that, for every $x(t) \in C$, $|x| < a$, $h(x(t), t) \in B$. Let there also exist a constant $\gamma > 0$ such that

$$|h(x', t) - h(x'', t)|_B \leq \gamma |x' - x''|$$

for every pair x' and $x'' \in C$, $|x'| < a$, $|x''| < a$.

Theorem 1. Let $\beta = |h(0, t)|_B$. Under the conditions of Lemma 3, if $K\gamma < 1$ and $\beta < K^{-1}(1 - K\gamma)a$, then for each $\xi_0 \in B_0$, $\|\xi_0\| < b = S^{-1}((1 - K\gamma)a - K\beta)$, there exists a unique solution $x(t)$ of equation (1) such that $|x| < a$ and $P_0 x(0) = \xi_0$; this solution satisfies the inequality

$$|x| \leq (1 - K\gamma)^{-1}(K\beta + S\|\xi_0\|). \tag{4}$$

The theorem is proved by the method of successive approximations on the basis of Lemmas 2 and 3.

From (4) follows the boundedness of solutions of equation (1).

Corollary. If $h(0, t) = 0$, then the zero solution of equation (1) is stable in the sense of Lyapunov.

Indeed, for $\beta = 0$, from (4) we have:

$$|x| \leq (1 - K\gamma)^{-1} S \|\xi_0\|.$$

Let us now consider the Cauchy problem

$$\frac{dx(t)}{dt} = A(t)x(t) + h(x(t), t), \qquad x(0) = x_0, \tag{5}$$

where the operator $A(t)$ satisfies the following conditions:

C_1. For each $t \in J$, $A(t)$ satisfies the condition

$$\|(I - \alpha A(t))^{-1}\| \leq 1$$

for $\alpha > 0$.

C_2. 1) The domain of definition of $A(t)$ does not depend on t; 2) $B(t,s) = (I - A(t))(I - A(s))^{-1}$ is uniformly bounded for all s, t; 3) $B(t,s)$ has bounded variation with respect to t, at least for some s.

As T. Kato ([1]) showed, under conditions C_1 and C_2 there exists $U(t,s)$ with certain properties, whence our conditions 1° and 2° follow.

Under conditions C_1 and C_2, we associate with the Cauchy problem (5) the nonlinear integral equation (1), whose solution we shall call a **generalized solution** of the Cauchy problem (5). Consequently, Theorem 1 establishes boundedness, and then stability, of generalized solutions of the Cauchy problem (5).

If the operator $A(t)$ also satisfies the condition that

$$A(t)\frac{d}{dt}A^{-1}(t)$$

is a strongly continuous operator in t; if $h(x(t),t)$ is a strongly differentiable function in t, and $x_0 \in D(A(t))$, then the generalized solution is strongly differentiable and is a solution of the Cauchy problem (5) ([2–4]).

Since nonstationary problems for a parabolic equation, in a certain sense, are reducible to the Cauchy problem (5), Theorem 1 gives boundedness (stability) conditions for solutions of the indicated nonstationary problems.

The question of stability of solutions of equation (5) was first investigated by M. G. Krein ([5]) in the case of bounded $A(t)$. Theorem 1 is a generalization of the theorem of Massera and Schäffer ([6]) for almost bounded $A(t)$, which in turn is a generalization of the well-known theorem of Perron ([7]) for finite-dimensional space.

Theorem 1 can also be formulated and proved for B distinct from C.

Institute of Mathematics and Mechanics
Academy of Sciences of the Azerbaijan SSR

Received
3 XII 1960

REFERENCES

[1] T. Kato, J. Math. Soc. Japan, **5** (1953); *Matematika*, **2**, 4 (1958).
[2] M. A. Krasnosel' skii, S. G. Krein, Tr. 3 Vsesoyuzn. matem. s" ezda, **3**, 1958, p. 73.
[3] T. Kato, Div. Electromag. Res. Inst. Math. Sci., N. Y. Univ., Res. Rep. No. BK-11 (1955).
[4] M. A. Krasnosel' skii, S. G. Krein, P. E. Sobolevskii, DAN, **111**, No. 1 (1956).
[5] M. G. Krein, UMN, **3**, No. 3 (1948).
[6] J. L. Massera, J. J. Schäffer, Ann. Math., **67**, No. 3, 517 (1958).
[7] O. Perron, Math. Zs., **32** (1930).