# MATHEMATICS

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1961

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**Abstract**

**Full Text**

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# ON THE ELEMENTARY THEORIES OF LOCALLY FREE UNIVERSAL ALGEBRAS

A set $A$ together with some sequence $\varphi_i(x_1, \dots, x_{r_i})$ $(i = 1, \dots, s)$ of operations defined on it is called an **algebra of type** $\langle r_1, \dots, r_s \rangle$. This type will henceforth be regarded as arbitrary, but fixed. The symbols $\varphi_1, \dots, \varphi_s$ will denote the fundamental operations of an arbitrary algebra of the given type. Object symbols $x, y, \dots$ will be called **terms of length 1**. If $a_1, \dots, a_{r_i}$ are terms of lengths $n_1, \dots, n_{r_i}$, then the expression $\varphi_i(a_1, \dots, a_{r_i})$ is called a **term of length** $n_1 + \dots + n_{r_i} + 1$. Let $a(a_1, \dots, a_n)$ be a term composed of elements $a_1, \dots, a_n$ of the algebra $A$ and of the functional symbols $\varphi_1, \dots, \varphi_s$. Performing on $a_1, \dots, a_n$ the operations indicated in the notation of the term, we obtain an element $a \in A$, called the **value** of the term $a$. The elements $a_1, \dots, a_n$ are **mutually free** in $A$ if the values of arbitrary terms $a, b$ in $a_1, \dots, a_n$ are equal only when the terms $a, b$ themselves are (graphically) equal. An algebra $A$ is called **free** if it has a mutually free system of generators. An algebra $A$ is **locally free** if each of its subalgebras generated by a finite system of elements is free. A free algebra is also locally free, since every subalgebra of a free algebra is free ([4]).

The class of locally free algebras is completely characterized by the axioms

$$\varphi_i(x_1, \dots, x_{r_i}) \neq \varphi_j(y_1, \dots, y_{r_j}) \qquad (i \neq j;\ i, j = 1, \dots, s); \tag{1}$$

$$\varphi_i(x_1, \dots, x_{r_i}) = \varphi_i(y_1, \dots, y_{r_i}) \to x_1 = y_1 \ \&\ \cdots \ \&\ x_{r_i} = y_{r_i} \quad (i = 1, \dots, s). \tag{2}$$

By formulas we shall henceforth mean formulas of the narrow predicate calculus (NPC), composed from equalities of the form $f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$, where $f, g$ are terms, by means of the logical operations $\neg$, $\&$, $\vee$, $\forall$, $\exists$.

The purpose of the present note is to indicate an algorithm which, for each formula $\mathfrak{F}(x_1, \dots, x_n)$, constructs a formula $\mathfrak{F}^*$ of a simpler form, equivalent to $\mathfrak{F}$ on every locally free algebra.

As a consequence it will follow that the set of closed formulas true on locally free algebras is algorithmically decidable.

An algebra with one binary operation is called a **groupoid**. In the works of Post ([2]) and Yas' kovskii ([3]) it was shown that the question of the realizability of any formula of NPC reduces to the question of the realizability of a suitable formula in the class of all free groupoids with an additional unary predicate, and thus the class of NPC formulas true on all free groupoids with a unary predicate is algorithmically undecidable. From the general result of the present note it follows that the class of formulas true on a free groupoid (without an additional predicate) is decidable.

We introduce the following permanent notation:

$$N_i(x) \leftrightarrow (y_1 \dots y_{r_i})(x \neq \varphi_i(y_1, \dots, y_{r_i})) \qquad (i = 1, \dots, s);$$

$$N_{\mathfrak{p}}(x) \leftrightarrow N_{i_1}(x) \& \dots \& N_{i_k}(x) \qquad (\mathfrak{p} = \langle i_1, \dots, i_k <;\ i_1 < \dots < i_k);$$

$$E_{\mathfrak{p}}^m \leftrightarrow (\exists y_1 \dots y_m)(N_{\mathfrak{p}}(y_1) \& \dots \& N_{\mathfrak{p}}(y_m) \& \bigwedge_{i \neq j} y_i \neq y_j) \qquad (m = 2, 3, \dots);$$

$$E_{\mathfrak{p}}^1 \leftrightarrow (\exists y) N_{\mathfrak{p}}(y); \qquad D_{\mathfrak{p}}^m \leftrightarrow \neg E_{\mathfrak{p}}^m \qquad (m = 1, 2, \dots).$$

Here $\bigwedge$, $\bigvee$ are the symbols, respectively, of conjunction and disjunction.

**Theorem.** *There exists an algorithm which, for every formula $\mathfrak{A}(x_1, \dots, x_n)$ with free individual variables $x_1, \dots, x_n$, constructs a formula $\mathfrak{A}^*(x_1, \dots, x_n)$, equivalent to $\mathfrak{A}$ on every locally free algebra and which is a conjunction of disjunctions of formulas of the form $N_i(z)$, $D_{\mathfrak{p}}^m$, $E_{\mathfrak{p}}^m$ and formulas of the form*

$$(\exists y_1 \dots y_m)(x_1 = f_1 \& \dots \& x_p = f_p \& \bigwedge x_{\alpha_i} \neq g_i \& \bigwedge y_{\beta_j} \neq h_j \& \bigwedge N_{\gamma_k}(y_{\delta_k})), \tag{3}$$

*where $f_l, g_i, h_j$ are terms in $x_{p+1}, \dots, x_n,\ y_1, \dots, y_m$; $\alpha_i = p + 1, \dots, n,\ \beta, \delta = 1, \dots, m$.*

If the formula $\mathfrak{A}$ contains no free individual variables, then, according to the theorem, it is equivalent to a conjunction of disjunctions of formulas $D_{\mathfrak{p}}^m$, $E_{\mathfrak{p}}^m$, asserting the existence or nonexistence in the algebra of a given number of elements of fixed type $N_{\mathfrak{p}}$. The number of types of elements is $2^s - 1$, and if for each type $N_{\mathfrak{p}}$ it is known how many distinct elements of this type there are in the algebra, then it is easy to compute also the value of the standard formula $\mathfrak{A}^*$.

Calling a standard class a class of algebras characterized by one of the axioms $D_{\mathfrak{p}}^m$ or $E_{\mathfrak{p}}^m$, we arrive at the conclusion that *every axiomatizable class of locally free algebras is an intersection of finite unions of standard classes.*

We shall now describe the algorithm for reducing a formula $\mathfrak{A}$ to the normal form $\mathfrak{A}^*$, whose existence is asserted by the theorem.

Let us agree to call an **∃-formula** a formula constructed from expressions of the form $f = g$, $f \neq g$, $N_i(z)$, where $f, g$ are terms, only by means of the symbols $\&, \vee, \exists$, i.e., without the symbols $\neg$ and $\forall$.

**Algorithm for reducing ∃-formulas.** Let a prenex form for $\mathfrak{A}$ be $(\exists y_1 \dots y_m)\mathfrak{A}_0$, where $\mathfrak{A}_0$ is the quantifier-free part ([1]). Reducing $\mathfrak{A}_0$ to disjunctive form and distributing the quantifiers $\exists$, we reduce the matter to transforming a formula $(\exists y_1 \dots y_m)\mathfrak{A}_1$, where $\mathfrak{A}_1$ is a conjunction of terms of the form $D^i_{\mathfrak{p}}$, $E^i_{\mathfrak{p}}$, $N_i(x)$, $N_i(y)$, $f = g$, $f \neq g$. All terms not depending on $y_1, \dots, y_m$ may be taken out and thereafter not considered. By virtue of (1), (2), terms of the form $\varphi_i(a_1, \dots, a_{r_i}) = \varphi_j(b_1, \dots, b_{r_j})$ for $i \neq j$ are false, and terms $\varphi_i(a_1, \dots, a_{r_i}) = \varphi_i(b_1, \dots, b_{r_i})$ may be replaced by the equivalent expression $a_1 = b_1 \& \dots \& a_{r_i} = b_{r_i}$. If in $\mathfrak{A}_1$ there is a term of the form $x = f$, then $x$ cannot occur in $f$, since otherwise this term will have the value false. Leaving such a term unchanged, in all the other terms in which $x$ occurs we replace $x$ by $f$. Repeating several times transformations of the indicated types, after a finite number of steps we arrive at a formula of the required form.

**Algorithm for reducing the negation of an ∃-formula.** Since an ∃-formula is reducible to a conjunction of disjunctions of standard formulas of the form (3) and $E^m_{\mathfrak{p}}$, $D^m_{\mathfrak{p}}$, in order to reduce the negation of an ∃-formula it is enough to be able to reduce the negations of these standard parts. But

$$\neg E^m_p \leftrightarrow D^m_p \quad \text{and} \quad \neg D^m_p \leftrightarrow E^m_p,$$

and $\neg N_i(x)$ has the form (3). Therefore it is necessary to be able to transform only the negation of formula (3), i.e., the formula

$$(y_1, \dots, y_m)\left(\bigwedge x_l = f_l \to \bigvee x_{\alpha_i} = g_i \vee \bigvee y_{\beta_j} = h_j \vee \bigvee \neg N_{i_k}(y_{\delta_k})\right).$$

This formula can be rewritten in the form

$$(y_{t+1} \dots y_m)\left(\bigwedge x_l = f_l \to (y_1 \dots y_t)\,\mathfrak{B}\right), \tag{4}$$

where $y_1, \dots, y_t$ are all those $y$' s that do not occur in the terms $f_1, \dots, f_p$. For given $x_1, \dots, x_p$, the system of equations $x_1 = f_1, \dots, x_p = f_p$ with respect to $y_{t+1}, \dots, y_m$ can have at most one solution. Therefore formula (4) is equivalent to the formula

$$(y_{t+1} \dots y_m)\left(\bigvee x_l \neq f_l\right) \vee (\exists y_{t+1} \dots y_m)\left(\bigwedge x_l = f_l \,\&\, (y_1 \dots y_t)\,\mathfrak{B}\right). \tag{5}$$

We have to transform into ∃-formulas the first member of the disjunction (5) and the expression $(y_1 \dots y_t)\mathfrak{B}$.

We shall illustrate the reduction of a formula of the form

$$(y_1 \dots y_\lambda)(x \neq f(x_1, \dots, x_\mu, y_1, \dots, y_\lambda))$$

by the example $(y)(x \neq x_1 y \cdot y)$, assuming that a groupoid is being considered.

Since for a given $x$ in a locally free groupoid the equation

$$x = uv \cdot w$$

has at most one solution for $u, v, w$, the indicated formula is equivalent to the formula

$$(\exists uv)(x = x_1 u \cdot v \,\&\, u \neq v) \vee (\exists uv)(x = uv \,\&\, (w)(u \neq x_1 w)),$$

where

$$(w)(u \neq x_1 w) \leftrightarrow (\exists yz)(u = yz \,\&\, y \neq x_1) \vee N(u).$$

Suppose that for the formula

$$(y_1 \dots y_\lambda)(x_1 \neq f_1 \vee \dots \vee x_q \neq f_q)$$

an equivalent $\exists$-form has already been found. The longer formula

$$(y_1 \dots y_\lambda)(x_1 \neq f_1 \vee \dots \vee x_{q+1} \neq f_{q+1})$$

is, for the reasons indicated above, equivalent to the formula

$$(y_1 \dots y_\lambda)(x_1 \neq f_1 \vee \dots \vee x_q \neq f_q) \vee (\exists y_1 \dots y_\sigma)(x_1 = f_1 \,\&\, \dots \,\&\, x_q = f_q \,\&\, (y_{\sigma+1} \dots y_\lambda)(x_{q+1} \neq f_{q+1})),$$

where $y_{\sigma+1}, \dots, y_\lambda$ are all those bound variables which do not occur explicitly in $f_1, \dots, f_q$. The reduction of both subformulas beginning with universal quantifiers was indicated above.

Now consider in (5) the subformula $(y_1 \dots y_t)\mathfrak{B}$. Its reduction is based on the following lemma.

**Lemma.** Let $T_1, \dots, T_t$ be fixed sets of elements of a locally free algebra. If each of these sets contains more than $u + v$ elements, then the expression

$$(\forall y_1 \in T_1) \dots (\forall y_t \in T_t)(x_{\alpha_1} = g_1 \vee \dots \vee x_{\alpha_u} = g_u \vee y_{\beta_1} = h_1 \vee \dots \vee y_{\beta_v} = h_v) \quad (6)$$

is false for any $x_1, \dots, x_n, y_{t+1}, \dots, y_m$, provided that, for every $i$, the term $g_i$ is distinct from $x_{\alpha_i}$ and the term $h_i$ is distinct from $y_{\beta_i}$. If the latter condition is not satisfied, then expression (6) is identically true.

Indeed, each of the equations $x_{\alpha_i} = g_i$ has, for those $y$' s which actually occur in it, at most one solution. Removing these solutions from the corresponding sets $T_1, \dots, T_t$, we obtain sets $T_1', \dots, T_t'$, each containing not fewer than $v$ elements; moreover, for ⋯

$y_1 \in T_1', \dots, y_t \in T_t'$, all the equalities $x_{\alpha_1} = g_1, \dots, x_{\alpha_u} = g_u$ will be false. Now fix $y_{\beta_1}$ in $T_{\beta_1}'$ and, instead of (6), consider the formula

$$(\forall y_1 \in T_1') \cdots (\forall y_t \in T_t')(y_{\beta_1} = h_1 \vee \cdots \vee y_{\beta_v} = h_v) \quad (\forall y_{\beta_1} \text{ is omitted}),$$

to which we apply the same arguments, and so on.

A locally free algebra is infinite. Therefore, if in $\mathfrak{B}$ there are no terms $\neg N_\gamma(y_s)$, then it follows from the lemma that the formula $(y_1 \dots y_t)\mathfrak{B}$ will be either identically false or identically true.

Suppose that in $\mathfrak{B}$ there are terms $\neg N_\gamma(y_s)$. If not all $y$' s occur in these terms, then, transforming $(y_1 \dots y_t)\mathfrak{B}$ to the form

$$(y_1 \dots y_w)\left(\bigvee \neg N_{\gamma_k}(y_{\delta_k}) \vee (y_{w+1} \dots y_t)\left(\bigvee x_{\alpha_i} = g_i \vee \bigvee y_{\beta_j} = h_j\right)\right), \tag{7}$$

we conclude, on the basis of the lemma, that the expression

$$(y_{w+1} \dots y_t)\left(\bigvee x_{\alpha_i} = g_i \vee \bigvee y_{\beta_j} = h_j\right)$$

is either identically true or identically false. In the second case (7) will be equivalent to a disjunction of the form

$$D_{\mathfrak{p}1}^1 \vee \cdots \vee D_{\mathfrak{p}v}^1,$$

and in the first case (7) is identically true.

It remains to consider formula (7) under the condition that $w = t$, i.e., that each of $y_1, \dots, y_t$ occurs in a suitable term $\neg N_\gamma(y_s)$. Combining the terms referring to one and the same $y_i$, we transform (7) to the form

$$(y_1 \dots y_t)\left(N_{\mathfrak{p}1}(y_1) \mathbin{\&} \cdots \mathbin{\&} N_{\mathfrak{p}t}(y_t) \to \bigvee x_{\alpha_i} = g_i \vee \bigvee y_{\beta_j} = h_j\right). \tag{8}$$

Let $M_i$ be the totality of all $N_{\mathfrak{p}i}$-elements of the algebra. Rewriting (8) in the form

$$\begin{aligned}(\forall y_1 \in M_1) \cdots (\forall y_t \in M_t)(x_{\alpha_1} = g_1 \vee \cdots \vee x_{\alpha_u} = g_u \vee y_{\beta_1} = h_1 \vee \cdots \\ \cdots \vee y_{\beta_v} = h_v),\end{aligned} \tag{9}$$

we conclude, on the basis of the lemma, that (9) is identically false if each set $M_i$ contains more than $u + v$ elements. Consequently, formula (8) is equivalent to the expression

$$(D_{\mathfrak{p}1}^{z+1} \mathbin{\&} \mathfrak{F}) \vee \dots \vee (D_{\mathfrak{p}t}^{z+1} \mathbin{\&} \mathfrak{F}) \quad (z = u + v),$$

where $\mathfrak{F}$ denotes formula (8) itself. But

$$D_{\mathfrak{p}}^{z+1} \leftrightarrow D_{\mathfrak{p}}^{1} \vee (D_{\mathfrak{p}}^{2} \mathbin{\&} E_{\mathfrak{p}}^{1}) \vee \dots \vee (D_{\mathfrak{p}}^{z+1} \mathbin{\&} E_{\mathfrak{p}}^{z}),$$

therefore formula (8) is equivalent to a disjunction of formulas $D_{\mathfrak{p}}^{1}$ and

$$D_{\mathfrak{p}}^{\sigma+1} \mathbin{\&} E_{\mathfrak{p}}^{\sigma} \mathbin{\&} \mathfrak{F} \quad (\sigma = 1, \dots, z).$$

But

$$D_{\mathfrak{p}1}^{\sigma+1} \mathbin{\&} E_{\mathfrak{p}1}^{\sigma} \mathbin{\&} \mathfrak{F} D \leftrightarrow$$

$$\leftrightarrow D_{\mathfrak{p}1}^{\sigma+1} \mathbin{\&} (\exists w_1 \dots w_\sigma) \left( N_{\mathfrak{p}1}(w_1) \mathbin{\&} \cdots \mathbin{\&} N_{\mathfrak{p}1}(w_\sigma) \mathbin{\&} \bigwedge w_i \neq w_j \mathbin{\&} \bigwedge \mathfrak{G}(u_i) \right),$$

where we have put

$$\mathfrak{G}(u_i) \leftrightarrow (y_2 \dots y_t) \left( N_{\mathfrak{p}2}(y_2) \mathbin{\&} \cdots \mathbin{\&} N_{\mathfrak{p}t}(y_t) \rightarrow \mathfrak{H}(u_i, y_2, \dots, y_m, x_1, \dots, x_n) \right),$$

$$\mathfrak{H}(y_1, \dots, y_m, x_1, \dots, x_n) \leftrightarrow \bigvee x_{\alpha_i} = g_i \vee \bigvee y_{\beta_j} = h_j.$$

Thus the matter has been reduced to bringing formulas $\mathfrak{G}(u_i)$ of the form (8), containing a smaller number of universal quantifiers, into the desired form.

Received
9 III 1961

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*Note: Figure translations are in progress. See original paper for figures.*