MATHEMATICS

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1961

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Abstract

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A NOTE ON THE SCHRÖDINGER EQUATION WITH A SINGULAR POTENTIAL

(Presented by Academician I. G. Petrovskii on 25 XI 1960)

1. In some problems of quantum mechanics it becomes necessary to consider a Schrödinger equation of the form

$$-\Delta\psi + \varepsilon\delta(x)\psi = E\psi, \tag{1}$$

where $\delta(x)$ is the Dirac δ-function.

The solution of equation (1) involves well-known difficulties connected with the fact that the expression

$$H = -\Delta + \varepsilon\delta(x) \tag{2}$$

is not an operator in Hilbert space: if $\psi(0) = 0$, then $H\psi = -\Delta\psi$; if $\psi(0) \neq 0$, then $H\psi$ does not belong to Hilbert space for any ψ.

The aim of the present note is to give mathematical meaning to physical papers devoted to equation (1) (see, for example, (^{1})).

2. We first reproduce the solution of equation (1) that occurs in physical papers. To this end, consider a family of kernels $u_N(x, y)$ such that

$$\lim_{N\to\infty} u_N(x, y) = \delta(x)\delta(y). \tag{3}$$

In addition, we shall also suppose the parameter ε to depend on N. We now replace equation (1) by the equation

$$-\Delta\psi + \varepsilon(N)\int u_N(x, y)\psi(y)\, d^3y = E\psi. \tag{N}$$

To solve equation (N), we perform the Fourier transform. As a result we obtain

$$p^2\widetilde{\psi} + \frac{\varepsilon(N)}{8\pi^3}\int \tilde{u}_N(p, q)\widetilde{\psi}(q)\, d^3q = E\widetilde{\psi}; \tag{$\widetilde{N}$}$$

$$\tilde{u}_N(p,q) = \int e^{i(qy-px)} u_N(x,y)\, d^3x\, d^3y.$$

The function $\tilde{u}_N(p,q)$ obviously satisfies the requirement

$$\lim_{N\to\infty} \tilde{u}_N(p,q) = 1. \tag{3'}$$

Since the family of kernels $u_N(x,y)$ has been chosen with the sole condition (3), or, what is the same, ($3'$), and the final result of the subsequent calculations should not depend on the choice of the family u_N, we choose u_N so that

$$\tilde{u}_N(p,q) = \chi_N(p)\chi_N(q); \qquad \chi_N(p) = \begin{cases} 1 & \text{for } p^2 < N^2, \\ 0 & \text{for } p^2 > N^2. \end{cases} \tag{4}$$

If one uses expression (4) for $\tilde{u}_N$, then equation ($\widetilde{N}$) is easy to solve. The eigenfunctions belonging to the continuous spectrum turn out to be equal to:

$$\widetilde{\psi}^+_N(p,s) = \delta(p-s) - \frac{\varepsilon'(N)\chi_N(p)\chi_N(s)}{1+\varepsilon'(N)\displaystyle\int \frac{\chi^2_N(p)\, d^3p}{p^2-s^2-i0}}\, \frac{1}{p^2-s^2-i0}. \tag{5}$$

$$s^2 = E, \qquad \varepsilon' = \frac{\varepsilon(N)}{8\pi^3}.$$

Further:

$$\int \frac{\chi^2_N(p)\, d^3p}{p^2-s^2-i0} = 4\pi \int_0^N \frac{p^2\, dp}{p^2-s^2-i0} = 4\pi \left(N + \frac{|s|}{2} \left(-\pi i + \ln \frac{N-|s|}{N+|s|} \right) \right). \tag{6}$$

From (5) and (6) it is clear that in order that the function $\widetilde{\psi}^+_N$ have a nontrivial limit as $N\to\infty$, one must set

$$\varepsilon'(N) = \frac{\alpha}{1-4\pi\alpha N},$$

where α is an arbitrary constant.

In this case the limit of $\widetilde{\psi}^+_N$ as $N\to\infty$ is equal to

$$\widetilde{\psi}^+ = \delta(p-s) - \frac{\alpha}{1-2\pi^2 i\alpha|s|}\, \frac{1}{p^2-s^2-i0}. \tag{7}$$

3. Consider the Fourier transform of expression (2):

$$\widetilde{H}\psi = p^2\psi + \varepsilon' \int \psi \, d^3p. \tag{8}$$

If the integral $\int \psi \, d^3p = 0$, then $\widetilde{H}\psi = p^2\psi$.

Denote by D_L the set of functions for which

$$\int p^4 |\psi|^2 \, d^3p < \infty, \qquad \int \psi \, d^3p = 0.$$

By L denote the operator, defined in D_L, of multiplication by p^2.

It turns out that the operator L is a closed symmetric operator with deficiency indices $(1, 1)$.

Using the general theory of extensions (see, for example, (2)), it is not difficult to construct all extensions of the operator L. It is easy to verify that all these extensions are given by the formula

$$H_\alpha \psi = p^2\psi + \lim_{N\to\infty} \frac{\alpha}{1 - 4\pi\alpha N} \int \chi_N(p)\psi(p) \, d^3p, \tag{9}$$

where $\psi(p)$ is a function possessing the properties:

$$\int \chi_N(p)\psi(p) \, d^3p = c(1 - 4\pi\alpha N) + o(1), \qquad \int |H_\alpha \psi|^2 \, d^3p < \infty. \tag{9'}$$

Let us note that for $\alpha \neq 0$ equality (9) may be replaced by

$$H_\alpha \psi = p^2\psi - \lim \frac{1}{4\pi N} \int \chi_N(p)\psi(p) \, d^3(p).$$

Thus, for $\alpha \neq 0$, the dependence of H_α on α appears only as the dependence on α of the domain of definition of H_α.

Let us verify that the eigenfunctions of the continuous spectrum of H_α are given by formula (7). For this purpose we construct the resolvent of the operator H_α. Let $(H_\alpha - z)f = g$. Then (2)

$$f(p) = \frac{g(p)}{p^2 - z} + \frac{M}{p^2 - z}. \tag{10}$$

In order to determine M, we use the fact that $f(p)$ satisfies condition $(9')$. From (10) we have:

$$\int f(p)\chi_N(p)\,d^3p = \int \frac{\chi_N(p)g(p)}{p^2-z}\,d^3p +$$

$$+ M\cdot 4\pi\left[N + \frac{\sqrt{z}}{2}\left(\pi i \operatorname{sign}\operatorname{Im}\sqrt{z} + \ln\frac{N-\sqrt{z}}{N+\sqrt{z}}\right)\right].$$

On the other hand, from (9′) we obtain that

$$\int f(p)\chi_N(p)\,d^3p = c(1-4\pi\alpha N) + o(1).$$

Comparing the last two expressions, we obtain, first, that $M = -\alpha c$, and, second, that

$$\int \frac{\chi_N(p)g(p)}{p^2-z}\,d^3p - 4\pi\alpha c\,\frac{\sqrt{z}}{2}\,\pi i \operatorname{sign}\operatorname{Im}\sqrt{z} - c \to 0.$$

Hence

$$c = \frac{\displaystyle\int \frac{g(p)\,d^3p}{p^2-z}}{1+2\pi^2\alpha i\sqrt{z}\operatorname{sign}\operatorname{Im}\sqrt{z}}.$$

Thus,

$$f(p) = \frac{g(p)}{p^2-z} - \frac{\alpha}{1+2\pi^2 i\alpha\sqrt{z}\operatorname{sign}\operatorname{Im}\sqrt{z}}\,\frac{1}{p^2-z}\int\frac{g(q)\,d^3q}{q^2-z}.$$

Hence we find the kernel of the resolvent

$$G(p,q,z) = \frac{\delta(p-q)}{p^2-z} - \frac{\alpha}{1+2\pi^2 i\alpha\sqrt{z}\operatorname{sign}\operatorname{Im}\sqrt{z}}\,\frac{1}{(p^2-z)(q^2-z)}. \qquad (11)$$

Using formula (11), it is easy to obtain the eigenfunctions of the continuous spectrum:

$$\psi_+(p,s) = \lim_{\varepsilon\to 0}\frac{1}{i}\,\varepsilon G(p,s,s^2+i\varepsilon).$$

Carrying out the limiting passage, we obtain expression (7) for the eigenfunction.

Using the expression for $\psi_+(p,s)$ and the analogous expression for ψ_-, one can construct the scattering operator by the formula $S(s_1,s_2) = \int \psi_+(p,s_1)\times\psi_-(p,s_2)\,d^3p$. As a result of the computations, one obtains the result indicated in (1).

4. It is not difficult to obtain an expression for H_α in the x-representation:

$$H_\alpha f = -\Delta f + \alpha \lim_{N\to\infty} \frac{1}{1-4\pi\alpha N} \frac{\sin N|x|}{|x|} \int \frac{\sin N|y|}{|y|} f(y)\, d^3y. \qquad (12)$$

The domain of definition of H_α consists of functions satisfying the condition

$$\int \frac{\sin N|x|}{|x|} f(x)\, d^3x = c(1-4\pi\varepsilon N) + o(1), \qquad \int |H_\alpha f|^2\, d^3x < \infty. \qquad (12’)$$

Thus, the mathematical content of the physicists' treatment of equation (1) consists in replacing expression (2) by the operator (12), ($12'$), which is an extension of the operator $-\Delta$ from the domain of definition consisting of functions $f(x)$ for which $f(0) = 0$, to the domain consisting of functions satisfying condition ($12'$).

Received
24 XI 1960

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[2] N. I. Akhiezer, I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, Moscow, 1950.

Note: Figure translations are in progress. See original paper for figures.