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Abstract

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Yu. L. Daletskii

CONTINUAL INTEGRALS ASSOCIATED WITH CERTAIN DIFFERENTIAL EQUATIONS AND SYSTEMS

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In the present note, the method set forth in (1) for representing solutions in the form of continual integrals is extended to a new class of differential equations.

1. Let $M(x_0, x_l)$ be the space of bounded vector-functions $x(t)$ $(0 \leqslant t \leqslant l)$ with values in the finite-dimensional space R_ν, satisfying the conditions $x(0) = x_0$, $x(l) = x_l$. Further, let q be a partition

$$0 = t_0 < t_1 < \dots < t_n < t_{n+1} = l;$$

R_ν^q the space of elements $(x_1, x_2, \dots, x_n)$, where $x_k \in R_\nu$, and $Q(q, R)$ the cylindrical set in $M(x_0, x_l)$ generated by the partition q and the Borel set R from R_ν^q. Suppose that on the sets $Q(q, R)$ a function $\mu(q, R)$ is defined, whose values are m-dimensional matrices and which is an abstract measure of bounded variation in each R_ν^q, satisfying the usual consistency conditions. Such a function $\mu_S(q, R)$ is constructed in the usual way (as in the theory of Markov processes) if a family of matrices $S(t_1, t_2; x_1, x_2)$ $(t_1 < t_2)$ is given, satisfying, for some measure $\sigma(x)$ in R_ν, the equation

$$\int_{R_\nu} S(t_2, t_3; x_2, x_3) S(t_1, t_2; x_1, x_2)\, d\sigma(x_2) = S(t_1, t_3; x_1, x_3). \tag{1}$$

If $\Phi(x(t))$ is a functional on $M(x_0, x_l)$, then put

$$I(\Phi) = \lim_q I_q(\Phi) = \lim_q \int_{R_\nu^q} \Phi(x_q(t))\, \mu(q, dx), \tag{2}$$

where $x_q(t) = x(t_{k-1})$ for $t_{k-1} \leqslant t < t_k$ $(k = 1, \dots, n+1)$ and $x_q(t) = x_l$ for $t = l$. If $I(\Phi)$ has meaning, then we shall call it a continual integral and shall denote it by

$$I(\Phi) = \int_{M(x_0, x_l)}^{*} \Phi(x(t))\, d\mu(x(t)). \tag{3}$$

If the function $\mu(q, R)$ has bounded variation, then it can be extended in the usual way to a certain countably additive measure. Integrals with respect to this measure, in contrast to (3), will be written without the asterisk. For bounded continuous functionals $\Phi(x)$, the two definitions coincide (2).

Theorem 1. If $S(t_1, t_2; x_1, x_2)$ is a real function continuous in x_1, x_2 and

$$\int_{R_\nu} S(t_1, t_2; x_1, x_2)\, d\sigma(x_2) \geqslant 1,$$

then the function $\mu_S(q, R)$ has bounded variation if and only if, in the formula written above, the equality sign holds and $S(t_1, t_2; x_1, x_2) \geqslant 0$.

2. Let $\mathfrak{H}$ be a Hilbert space; T a self-adjoint operator with bounded inverse; $D = D_T$ the space of fundamental elements constructed from T; N the corresponding space of generalized elements (1).

Consider the solution $U(\tau, t)$ of the operator equation $dU/dt = A(t)U$ with the condition $U(\tau, \tau) = I$, assuming that the conditions under which it exists and is well-defined are satisfied [3–6]. Suppose that for some $p > 0$, $\|TA^{-p}\| < \infty$. Then the expressions

$$S_{jk}(t_1, t_2; x_1, x_2) = (U(t_1, t_2)\xi_{jx_1}, \xi_{kx_2}),$$

where ξ_{kx} $(k = 1, \dots, m;\ x \in R_\nu)$ is a complete family of generalized elements, are meaningful, and the matrix $S = \|S_{jk}\|$ satisfies condition (1).

Theorem 2. Let $m = 1$ and

$$S(t_1, t_2; x_1, x_2) = (e^{A(t_2 - t_1)}\xi_{x_1}, \xi_{x_2})$$

$(A = \text{const})$. Suppose there exists a sequence $\varphi_n \in D$ such that $A\varphi_n \in D$, and the sequences of functions $\varphi_n(x) = (\varphi_n, \xi_x)$ and $\psi_n(x) = (A\varphi_n, \xi_x)$ are uniformly bounded, with $\varphi_n(x) \to 1$, $\psi_n(x) \to 0$ as $n \to \infty$ almost everywhere in R_ν with respect to $\sigma(x)$. Then, if

$$\int_{R_\nu} |S(t_1, t_2; x_1, x_2)|\, d\sigma(x_2) < \infty$$

for $t_2 > t_1$, then

$$\int_{R_\nu} S(t_1, t_2; x_1, x_2)\, d\sigma(x_2) = 1.$$

An analogous theorem also holds for $m > 1$ and for A depending on t.

3. Let A be a self-adjoint negative definite operator, and let B be some generating operator such that $A + B$ is the generating operator of a semigroup [6] that maps $\mathfrak{H}$ into D. In [1], under the boundedness condition on the operators B and $A^{pBA^{-p}}$, the equality

$$e^{(A+B)t}\xi = \lim_q \sum_{k=1}^{n} e^{A(t_k - t_{k-1})} e^{B(t_k - t_{k-1})}\xi \quad (\xi \in N), \tag{4}$$

is established, where the limit is understood in the sense of strong convergence in D.

Theorem 3. Relation (4) is valid if, for some $\gamma > 0$ and $p_1 \geq p$, the conditions

$$\|A^{pBA_1^{-p}}\| < \infty, \qquad \|A^{p_1} e^{Bt} A^{-p_1}\| \leq e^{\gamma t}$$

are satisfied.

Let us introduce the fundamental matrix of the operator equation $d\psi/dt = (A+B)\psi$, corresponding to the chosen complete system of generalized elements:

$$W(t;x,y) = \left\|(e^{(A+B)t}\xi_{ix}, \xi_{jy})\right\|_{i,j=1}^{m}.$$

From formula (4) follows the representation

$$W(t;x_0,x_l) = \lim_q \int_{R_\nu^q} \sum_{r=1}^{n+1} G(t_r - t_{r-1}; x_{r-1}, x_r)\, d\sigma(x_1) \cdots d\sigma(x_n), \tag{5}$$

where

$$G_{ij}(t;x,y) = (e^{At} e^{Bt} \xi_{ix}, \xi_{jy}).$$

We note that these results also carry over to the case when the operators A and B depend on t.

4. Let us consider some special cases.

1) Let $B = V(C)$ be a bounded scalar function of a self-adjoint operator C, satisfying conditions similar to those considered in (1). Under these conditions, as is known ([7]), N contains a complete family of generalized eigen-elements of the operator C. From formula (5), for the fundamental matrix corresponding to this system, there follows the representation

$$W(t;x_0,x_l) = \int_{M(x_0,x_l)}^{*} \exp\left[\int_0^t V(x(t))\, dt\right] d\mu_S(x(t)), \tag{6}$$

where

$$S_{ij}(t;x,y) = (e^{At}\xi_{ix}, \xi_{jy}).$$

Formula (6) can be used to represent the fundamental matrix of the system

$$\partial\psi/\partial t = L(\psi) + B(x,t)\psi, \tag{7}$$

where ψ is an m-dimensional vector, $L(\psi)$ is a self-adjoint strongly elliptic system of differential operators, and $B(x,t)$ is a Hermitian matrix.

2) Let

$$B\xi_{jx} = \sum_{k=1}^{m} \beta_{jk}(x)\xi_{kx}$$

and

$$(e^{At}\xi_{jx}, \xi_{ky}) = \delta_{jk}(e^{At}\xi_{1x}, \xi_{1y}).$$

Then

$$W(l; x_0, x_l) = \int^*_{M(x_0,x_l)} \exp\left[\int_0^l \beta(x(t))\,dt\right] d\mu_S(x(t)),$$

where

$$S(t; x, y) = (e^{At}\xi_{1x}, \xi_{1y})$$

is a scalar function, and under the sign of the continual integral there stands a multiplicative integral. This formula can be applied to a system of the form (7), in which $L(\psi) = L_1(\psi) \cdot I$, where $L_1(\psi)$ is an elliptic operator and I is the identity matrix.

3) For simplicity consider the case $m = 1$, and let

$$e^{Bt}\xi_x = e^{t\beta(x,t)}\xi_{\gamma(x,t)},$$

where $\beta(x,t)$ is a scalar function and $\gamma(x,t)$ is a function with values in R_ν. Formula (5) can then be brought to the form

$$W(l; x_0, x_l) =$$

$$= \lim_b \int_{R_q^\nu} \exp\left\{\sum_{r=1}^{n+1} [\Delta t_r \beta(x_{r-1}, \Delta t_r) + f(\Delta t_r; x_{r-1}, x_r)]\right\} d\mu_S(q, R). \quad (8)$$

where

$$S(t; x, y) = (e^{At}\xi_x, \xi_y)$$

and

$$f(t; x, y) = \ln\frac{S(t, \gamma(x,y), y)}{S(t, x, y)}.$$

Let there exist a functional $\Phi(x(t))$ in $M(x_0, x_l)$ for which the value $\Phi(x_q(t))$ coincides with the expression under the sign exp in (8). Then

$$W(l; x_0, x_l) = \int^*_{M(x_0,x_l)} e^{\Phi(x(t))} d\mu(x(t)). \quad (9)$$

Carrying out analogous considerations in the case $m \geqslant 1$, one can obtain a representation of the fundamental matrix of a system of the form

$$\frac{\partial\psi}{\partial t} = L(\psi) \cdot I + \sum_{k=1}^{\nu} a_k(x,t)\frac{\partial\psi}{\partial x^{(k)}} + V(x,t)\psi, \quad (10)$$

where $I(\psi)$ is a self-adjoint elliptic operator, and $a_k(x,t)$ and $V(x,t)$ are certain matrices.

Theorem 4. *Let, in system (10),* $L(\psi) = a\Delta\psi$ *and* $a_k = U_k \Lambda_{kU} k^{-1}$, *where* Λ_k *are diagonal matrices. Then*

$$W(l; x_0, x_l) =$$
$$= \int^*_{M(x_0,x_l)} \exp\left\{ -\frac{1}{2a}\int_0^l \sum_{k=1}^{\nu} a_k\, dx^{(k)} - \frac{1}{4a}\int_0^l \sum_{k=1}^{\nu} (a_k^2 - U_k^1 U_k^{-1} a_k)\, dt - \int_0^l \operatorname{div} a(x)\, dt \right\} \times$$
$$\times d\mu(x(t)). \tag{11}$$

In this connection, for a functional of the form

$$\Psi(x(t)) = \int_0^l f(x, \dot{t})\, dx(t)$$

we set

$$\Psi(x_q(t)) = \sum_{r=1}^{n+1} f(x_{r-1}, t_{r-1})(x_r - x_{r-1}). \tag{12}$$

Remark 1. It can be shown that under certain modifications of formula (12), formula (11) will also change.* For example, if in (12) one takes $f(x_r, t_r)$ instead of $f(x_{r-1}, t_{r-1})$, then in (11) it is necessary to discard the last

* See, in this connection, the remark by R. Feynman ((8), p. 185).

term in the exponent; and if one takes $\frac{1}{2}[f(x_r, t_r) - f(x_{r-1}, t_{r-1})]$, then one must discard half of this term.

Remark 2. In the scalar case $U_k' = 0$ (this is also true for constant matrices a_k). Formula (11) in this case was obtained by a heuristic argument in (9). However, the method used there leads to an inaccuracy connected with the fact that the meaning of the expression $\Psi(x_q(t))$, considered above, is not indicated. It turns out that, in order to obtain the formula derived in (9), one must take for $\Psi(x_q(t))$ the expression indicated at the end of Remark 1; but then the continual integral in (11) cannot be regarded as a Lebesgue integral with respect to Wiener measure (see Theorem 5 below).

5. In the case when $L(\psi)$ in equation (10) is a second-order elliptic operator, the fundamental solutions $S(t; x, y)$ of the equation $\partial\psi/\partial t = L(\psi)$, as is known [10], are nonnegative, and therefore the set function $\mu_S(q, R)$ generates a measure $\mu(x(t))$ in the space $M(x_0, x_l)$.

Theorem 5. *Consider equation (10) in the scalar case. If in the representation (9) of its fundamental solution the relation* $\lim_q \Phi(x_q(t)) = \Phi(x(t))$ *holds, where the limit is understood in the mean square with respect to the measure* $\mu(x(t))$, *then the integral in this representation may be regarded as a Lebesgue integral with respect to the measure* $\mu(x(t))$.

Proof. The theorem is applicable in the case when the operator $L(\psi)$ is the generator of a Markov process for which stochastic integrals exist (see [11], p. 392).

In particular, this holds when $L(\psi) = a\Delta\psi$, and therefore in formula (11) in the scalar case the asterisk over the integral may be omitted.

For the case $\nu = 1$, the representability in the form of a Lebesgue integral of the fundamental solution of equation (10) (scalar) was shown by other methods in [12]. In the case $\nu > 1$, similar results follow from [13].

6. By a well-known method due to R. Feynman [8], one can show the representability, in the form of a limit of continual integrals, of the fundamental solution of an equation of the form $\partial\psi/\partial t = iL(\psi) + L_1(\psi)$. This device consists in considering the equation

$$\partial\psi/\partial t = (i + \varepsilon)L(\psi) + L_1(\psi),$$

   to which the results set forth above are applicable (for more detail see [1]), and in subsequently passing to the limit as $\varepsilon \to 0$. In this way, for example, one can obtain a representation of the fundamental solutions of an equation of Pauli type in a form analogous to (11).

We note that the set function $\mu(q, R)$ generated by the operator $(i + \varepsilon)L(\psi)$ is complex, and therefore Theorem 1 is not applicable to it. However, in this case as well, under certain conditions one can prove an analogous theorem. In particular, one can show that the function $\mu_S(q, R)$, constructed from the fundamental solution

$$S(t; x, y) = (4\pi Dt)^{1/2} e^{-x^2/4Dt}$$

of the equation $\partial\psi/\partial t = D\partial^2\psi/\partial x^2$ for $\operatorname{Im} D \neq 0$, is not of bounded variation. In this connection, the assertion made in [9] (p. 93) that this equation generates a measure just as good as the Wiener measure seems to us unjustified.

Kyiv Polytechnic Institute

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Note: Figure translations are in progress. See original paper for figures.